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Star products: a group-theoretical point of view

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Abstract

Adopting a purely group-theoretical point of view, we consider the star product of functions which is associated, in a natural way, with a square integrable (in general, projective) representation of a locally compact group. Next, we show that for this (implicitly defined) star product, explicit formulas can be provided. Two significant examples are studied in detail: the group of translations on phase space and the one-dimensional affine group. The study of the first example leads to the Groenewold–Moyal star product. In the second example, the link with wavelet analysis is clarified.

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1. Introduction

The concept of star product of functions is a remarkable achievement of theoretical physics. The archetype—and still nowadays, the most important realization—of this concept is the Groenewold–Moyal star product [1, 2] (see also the recent book [3] and references therein). Although there is no unique general mathematical framework encompassing all known star products, one can certainly single out a simple leading idea to which the various possible definitions of star products are more or less inspired: to replace the ordinary pointwise product of ($\mathbb{C}$-valued) functions defined on a certain set (a 'phase space' endowed with some structures: a differentiable manifold, a measure space, etc) with a suitable non-commutative, associative product that mimics the typical non-commutative behavior of linear operators.

We will make no attempt at surveying the rich and varied literature on star products. We will content ourselves with recalling that both algebraic–analytic [4–6] and differential–geometric [7–10] approaches to the subject have been adopted, also in view of different purposes and applications. It is also worth mentioning the fact that the most important topics where the formalism of star products plays a relevant role are, probably, the construction of quantum mechanics 'on phase space' and the study of the classical limit of quantum mechanics [3, 11, 12] (see also Kubo's seminal paper [13]). Thus, one may regard Wigner [14] and Weyl [15] as the fathers of this formalism.

More recently, a general approach to star products based on the idea of using suitable 'quantizers' and 'dequantizers' has been proposed and developed by various authors [16–22]. This approach is very close to applications in quantum mechanics since the star products of functions that one obtains are, by construction, nothing but the 'images' of the products of quantum-mechanical operators.

In our present contribution, we will adopt a purely group-theoretical point of view which is conceptually similar to the 'quantizer–dequantizer' approach cited above. Indeed, rather than trying to define a star product directly in a given space of functions (as usual, for instance, in the differential-geometric approach), we consider the star product (implicitly) induced by a suitable group-theoretical quantization–dequantization scheme. Clearly, at this point, the real problem is to find explicit formulas for the implicitly defined star product.

Before illustrating the main points of our work, it is worth mentioning that recently another group-theoretical approach to star products—in the context of a suitable quantization–dequantization scheme—has been elaborated [23]. However, this approach, differently from the approach adopted in the present paper, relies on the concept of 'frame transform' and it is not directly related to the Groenewold–Moyal product.

Let us now briefly outline our method and our main results. First, we show that by means of the quantization (Weyl) and dequantization (Wigner) maps generated by a square integrable (in general, projective) representation U of a locally compact group G—see [23–25]—it is possible to introduce, in a natural way, a star product in the Hilbert space $\mathrm{L}^2(G)$ of square integrable $\mathbb{C}$-valued functions on G. The product of two functions is obtained by quantizing them, by forming the product of the two operators thus obtained and, finally, by dequantizing this product. Endowed with the operation just described, $\mathrm{L}^2(G)$ becomes a H^*-algebra. We will then prove—this is the main result of the paper—that the star product in $\mathrm{L}^2(G)$ admits a simple explicit formula. More precisely, we will show that with every orthonormal basis in the Hilbert space of the representation U is associated a formula for the star product (however, all these formulas share the same general form). This basic result can be generalized or specialized in various ways. For instance, an expression of the '$\hat{K}$-deformed star product'—see [19, 20]—which is an interesting generalization of the star product, can also be obtained. On the other hand, in the case where G is unimodular, a particularly simple formula for the star product—a sort of 'twisted convolution' *à la* Grossmann–Loupias–Stein [4]—can be derived.

We believe that the point of view on star products adopted in this paper is very close to the 'original spirit' of the Groenewold–Moyal star product since it solely relies on (generalized) Wigner and Weyl maps. In fact, 'our' star product is *essentially* the Groenewold–Moyal star product in the case where the group G is the group of translations on phase space, i.e. the two products—the twisted convolution and the Groenewold–Moyal product—are related by the symplectic Fourier transform.

We stress that our approach relies on the *existence* of a square integrable representation U of the locally compact group G for defining an associated star product in $\mathrm{L}^2(G)$. This feature, however, should not be regarded as a limit of this approach. As is well known, when dealing with mathematics nothing is free: the weaker the assumptions, the poorer will be the results that one is able to prove. Moreover, our group-theoretical point of view is very natural having in mind applications to physics. If G is regarded as a 'symmetry group' of a quantum system and U as the symmetry action of this group in the Hilbert space $\mathcal{H}$ of the system, then the associated star product in $\mathrm{L}^2(G)$ is nothing but the realization in terms of functions of the product of quantum-mechanical operators (observables or states); moreover, it turns out that the star product is 'equivariant' with respect to the natural action of the symmetry group. Namely, the natural action of G on operators in $\mathcal{H}$ translates into (i.e. is intertwined by the dequantization map with) a simple transformation of the corresponding functions in $\mathrm{L}^2(G)$, and the star product of two *transformed* functions coincides with the *transformed* product of the two *untransformed* functions.

The paper is organized as follows. In section 2, we fix the main notations and we briefly recall some mathematical notions; in particular, we review some basic facts concerning square integrable representations. Next, in section 3, we define the dequantization (Wigner) and quantization (Weyl) maps generated by a square integrable representation, and we derive the relevant 'intertwining properties' of the Wigner map. On the basis of these definitions we then introduce—see section 4—the notion of star product associated with a square integrable representation, and we study its main properties. The star product introduced in such a way is, however, only *implicitly* defined. As already mentioned, it is a remarkable fact that it admits an *explicit* realization; furthermore, in the case of a unimodular group, a particularly simple formula can be derived. These results—that form the *core* of our paper—are stated and proved in section 5. In section 6, we consider two significant examples: the group of translations on phase space—which is related to the standard Groenewold–Moyal star product—and the affine group, which plays a central role in wavelet analysis. Finally, in section 7, a few conclusions are drawn.

2. Some known facts and notations

In this section, we will recall some basic facts of the theory of representations of topological groups; standard references on the subject are [26, 27]. We will also fix the main notations that will be used in the following sections.

Let G be a locally compact, second countable, Hausdorff topological group (in short, l.c.s.c. group). We will denote by μ_G and Δ_G, respectively, a *left Haar measure* (of course uniquely defined up to a multiplicative constant) and the *modular function* on G. The symbol e will indicate the unit element in G.

For the scalar product $\langle\cdot,\cdot\rangle$ in a separable complex Hilbert space $\mathcal{H}$, we will always follow the convention that it is linear in the *second* argument. The symbol $\mathcal{U}(\mathcal{H})$ will denote the *unitary group* of $\mathcal{H}$—i.e. the group of all unitary operators in $\mathcal{H}$—which, endowed with the strong operator topology, is a metrizable, second countable, Hausdorff topological group.

In the following, we will consider a weakly Borel[1] *projective representation* $U : G \to \mathcal{U}(\mathcal{H})$ of a l.c.s.c. group G in a separable complex Hilbert space $\mathcal{H}$—see [26], chapter VII—with *multiplier* $\mathtt{m}$:

$$U(e) = I, \qquad U(gh) = \mathtt{m}(g,h)\,U(g)\,U(h), \qquad \forall\, g,h \in G, \tag{2.1}$$

where I is the identity operator in $\mathcal{H}$. The multiplier $\mathtt{m} : G \times G \to \mathbb{T}$—with $\mathbb{T}$ denoting the circle group, i.e. the group of complex numbers of modulus one—is a Borel function satisfying the following conditions:

$$\mathtt{m}(g,e) = \mathtt{m}(e,g) = 1, \qquad \forall\, g \in G, \tag{2.2}$$

and

$$\mathtt{m}(g_1, g_2 g_3)\,\mathtt{m}(g_2, g_3) = \mathtt{m}(g_1 g_2, g_3)\,\mathtt{m}(g_1, g_2), \qquad \forall\, g_1, g_2, g_3 \in G. \tag{2.3}$$

[1] Namely, $G \ni g \mapsto \langle\phi, U(g)\,\psi\rangle \in \mathbb{C}$ is a Borel function, for any pair of vectors $\phi, \psi \in \mathcal{H}$. Projective representations (in particular, unitary representations) will always be implicitly assumed to be weakly Borel.

It is, moreover, immediate to check that $\mathtt{m}(g, g^{-1}) = \mathtt{m}(g^{-1}, g)$. Of course, in the case where $\mathtt{m} \equiv 1$, U is a standard unitary representation and, according to a well-known result, in this case the hypothesis that U is a weakly Borel map implies that it is strongly continuous.

We can identify the unitary dual of G with any (suitably topologized) maximal set of mutually unitarily inequivalent, irreducible (strongly continuous) unitary representations of G. We will denote by $\check{G}$ such a set, and we will call it *a realization of the unitary dual of* G. Recall that, if G is compact, then $\check{G}$ must be a finite or countable set. It is worth stressing that we will regard as compact groups all the finite groups (endowed with the discrete topology).

Assume that the projective representation $U : G \to \mathcal{U}(\mathcal{H})$ is *irreducible*. Given two vectors $\psi, \phi \in \mathcal{H}$, we define the function (called 'coefficient' of the representation U)

$$\mathsf{c}^U_{\psi,\phi} : G \ni g \mapsto \langle U(g)\,\psi, \phi\rangle \in \mathbb{C}, \tag{2.4}$$

and we consider the set of 'admissible vectors for U', i.e. $\mathsf{A}(U) := \big\{\psi \in \mathcal{H} \,|\, \exists\phi \in \mathcal{H} : \phi \neq 0,\ \mathsf{c}^U_{\psi,\phi} \in \mathrm{L}^2(G)\big\}$, where $\mathrm{L}^2(G) \equiv \mathrm{L}^2(G, \mu_G; \mathbb{C})$ (we will denote by $\langle\cdot,\cdot\rangle_{\mathrm{L}^2}$ and $\|\cdot\|_{\mathrm{L}^2}$ the scalar product and the norm in $\mathrm{L}^2(G)$). The representation U is said to be *square integrable* if $\mathsf{A}(U) \neq \{0\}$. Square integrable projective representations are characterized by the following result—see [28]—which is a generalization of a classical theorem of Duflo and Moore [29] concerning unitary representations.

Theorem 2.1. *Let the projective representation $U : G \to \mathcal{U}(\mathcal{H})$ be square integrable. Then, the set $\mathsf{A}(U)$ is a dense linear span in $\mathcal{H}$, stable under the action of U, and, for any pair of vectors $\phi \in \mathcal{H}$ and $\psi \in \mathsf{A}(U)$, the coefficient $\mathsf{c}^U_{\psi,\phi}$ is square integrable with respect to the left Haar measure μ_G on G. Moreover, there exists a unique positive self-adjoint, injective linear operator $\hat{D}_U$ in $\mathcal{H}$—the 'Duflo–Moore operator associated with U'—such that $\mathsf{A}(U) = \mathrm{Dom}(\hat{D}_U)$ and the following 'orthogonality relations' hold:*

$$\big\langle \mathsf{c}^U_{\psi_1,\phi_1}, \mathsf{c}^U_{\psi_2,\phi_2}\big\rangle_{\mathrm{L}^2} = \langle\phi_1, \phi_2\rangle\, \langle\hat{D}_U\,\psi_2, \hat{D}_U\,\psi_1\rangle, \tag{2.5}$$

for all $\phi_1, \phi_2 \in \mathcal{H}$ and all $\psi_1, \psi_2 \in \mathsf{A}(U)$. The Duflo–Moore operator $\hat{D}_U$ satisfies the relation

$$U(g)\,\hat{D}_U = \Delta_G(g)^{\frac{1}{2}}\,\hat{D}_U\,U(g), \qquad \forall\, g \in G; \tag{2.6}$$

it is bounded if and only if G is unimodular (i.e. $\Delta_G \equiv 1$) and, in such case, it is a multiple of the identity.

Remark 2.1. Let the representation U be square integrable. If the Haar measure μ_G is rescaled by a positive constant, then the Duflo–Moore operator $\hat{D}_U$ is rescaled by the square root of this constant. Thus, we will say that $\hat{D}_U$ is *normalized according to* μ_G. On the other hand, if a normalization of the left Haar measure on G is not fixed, $\hat{D}_U$ is defined up to a positive factor and we will call a specific choice a *normalization of the Duflo–Moore operator*. In particular, if G is unimodular, then $\hat{D}_U = I$ is a normalization of the Duflo–Moore operator, and the corresponding Haar measure will be said to be *normalized in agreement with* U. Moreover, the operator $\hat{D}_U$, being injective and positive self-adjoint, has a positive self-adjoint, densely defined inverse. As a consequence of (2.6), the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-1}\big) = \mathrm{Ran}(\hat{D}_U)$—like $\mathsf{A}(U) = \mathrm{Dom}(\hat{D}_U)$—is stable under the action of U and

$$U(g)^{-1}\,\hat{D}_U^{-1} = \Delta_G(g)^{\frac{1}{2}}\,\hat{D}_U^{-1} U(g)^{-1}, \qquad \forall\, g \in G. \tag{2.7}$$

From this relation, using the fact that $U(g)^{-1} = \mathtt{m}(g, g^{-1})\,U(g^{-1})$, we obtain

$$U(g)\hat{D}_U^{-1} = \Delta_G(g)^{-\frac{1}{2}}\,\hat{D}_U^{-1} U(g), \qquad \forall\, g \in G. \tag{2.8}$$

We finally note that, in the case where G is *not* unimodular, a square integrable representation of G cannot be finite-dimensional (since the associated Duflo–Moore operator is unbounded).

Let us recall a few other facts about square integrable representations:

(1) If the representation U of G is square integrable, then the orthogonality relations (2.5) imply that, for every nonzero admissible vector $\psi \in \mathsf{A}(U)$, one can define the linear operator

$$\mathfrak{W}_U^{\psi} : \mathcal{H} \ni \phi \mapsto \|\hat{D}_U\, \psi\|^{-1}\, \mathsf{c}^U_{\psi,\phi} \in \mathrm{L}^2(G) \tag{2.9}$$

—sometimes called *(generalized) wavelet transform* generated by U, with *analyzing vector* ψ—which is an isometry. The ordinary wavelet transform arises in the special case where G is the one-dimensional affine group $\mathbb{R} \rtimes \mathbb{R}^+_*$ (see [30, 31]); we will clarify this point in section 6. The isometry $\mathfrak{W}_U^{\psi}$ intertwines the representation U with the *left regular* $\mathtt{m}$*-representation* $R_{\mathtt{m}}$ of G in $\mathrm{L}^2(G)$, see [28], which is the projective representation (with multiplier $\mathtt{m}$) defined by

$$(R_{\mathtt{m}}(g) f)(g') := \vec{\mathtt{m}}(g, g')\, f(g^{-1}g'), \qquad g, g' \in G, \qquad f \in \mathrm{L}^2(G), \tag{2.10}$$

where $\vec{\mathtt{m}}(g, g') := \mathtt{m}(g, g^{-1})^* \,\mathtt{m}(g^{-1}, g')$, namely

$$\mathfrak{W}_U^{\psi}\, U(g) = R_{\mathtt{m}}(g)\, \mathfrak{W}_U^{\psi}, \qquad \forall\, g \in G. \tag{2.11}$$

Hence, U is unitarily equivalent to a subrepresentation of $R_{\mathtt{m}}$. Note that, for $\mathtt{m} \equiv 1$, $R \equiv R_{\mathtt{m}}$ is the standard left regular representation of G.

(2) Let the group G be compact (hence, unimodular), and let $\check{G}$ be a realization of the unitary dual of G. In this case, the irreducible *unitary* representations of G are finite-dimensional—we will denote by $\delta(U)$ the dimension of the Hilbert space $\mathcal{H} \equiv \mathcal{H}(U)$ of a representation $U \in \check{G}$—and square integrable (since the Haar measure on G is finite and the coefficients of these representations are a bounded functions). According to the Peter–Weyl theorem [27, 32], the Hilbert space $\mathrm{L}^2(G)$ admits the orthogonal sum decomposition

$$\mathrm{L}^2(G) = \bigoplus_{U \in \check{G}} \mathrm{L}^2(G)_{[U]}, \tag{2.12}$$

where $\mathrm{L}^2(G)_{[U]}$ is a finite-dimensional subspace of $\mathrm{L}^2(G)$—depending only on the unitary equivalence class $[U]$ of the representation U—that is characterized as follows:

- for every orthonormal basis $\{\chi_n\}_{n=1}^{\delta(U)}$ in the Hilbert space of the representation $U \in \check{G}$,

$$\mathrm{L}^2(G)_{[U]} = \bigoplus_{n=1}^{\delta(U)} \mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big); \tag{2.13}$$

  hence, $\dim(\mathrm{L}^2(G)_{[U]}) = \delta(U)^2$;
- for every $n \in \{1, \ldots, \delta(U)\}$, $\mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big)$ is an invariant subspace for the left regular representation R of G, and the restriction of R to $\mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big)$ is irreducible and unitarily equivalent to U.

Therefore, each representation $U \in \check{G}$ 'occurs with multiplicity $\delta(U)$ in the left regular representation R', i.e. R is unitarily equivalent to the representation

$$\bigoplus_{U \in \check{G}} \overbrace{U \oplus \cdots \oplus U}^{\delta(U)}. \tag{2.14}$$

Assuming that the Haar measure μ_G is normalized as usual for compact groups—i.e. that $\mu_G(G) = 1$—we have

$$\delta(U) \int_G \langle \phi_1, U(g)\, \psi_1\rangle\, \langle U(g)\, \psi_2, \phi_2\rangle \,\mathrm{d}\mu_G(g) = \langle \phi_1, \phi_2\rangle\, \langle \psi_2, \psi_1\rangle, \tag{2.15}$$

for all vectors $\phi_1, \psi_1, \phi_2, \psi_2 \in \mathcal{H}$. Hence, the Duflo–Moore operator associated with the square integrable representation U is of the form $d_U\, I$, where $d_U = \delta(U)^{-\frac{1}{2}}$.

(3) Denoting by $\hat{q}$, $\hat{p}$ the standard position and momentum operators in $\mathrm{L}^2(\mathbb{R})$, the map

$$U : \mathbb{R} \times \mathbb{R} \ni (q, p) \mapsto \exp(\mathrm{i}(p\hat{q} - q\hat{p})) \in \mathcal{U}(\mathrm{L}^2(\mathbb{R})) \tag{2.16}$$

is a projective representation of the (additive) group $\mathbb{R} \times \mathbb{R}$. This representation is square integrable and, fixing $(2\pi)^{-1}\,\mathrm{d}q\,\mathrm{d}p$ as the Haar measure on $\mathbb{R} \times \mathbb{R}$, we have that $\hat{D}_U = I$; see [23]. Therefore, the Haar measure $(2\pi)^{-1}\mathrm{d}q\mathrm{d}p$ is normalized in agreement with U. If $\psi_0 \in \mathrm{L}^2(\mathbb{R})$ is the ground state of the quantum harmonic oscillator, then $\{U(q, p)\,\psi_0\}_{q,p\in\mathbb{R}}$ is the family of standard *coherent states* [33, 34].

We conclude this section fixing some further notations and recalling a technical result. The symbol $\overline{\hat{C}}$ will indicate the closure of a closable operator $\hat{C}$ in $\mathcal{H}$. Given a subspace $\mathcal{S}$ of $\mathcal{H}$, we will denote by $\mathcal{S}^{\perp}$ the orthogonal complement of $\mathcal{S}$ in $\mathcal{H}$. We will denote by $\mathcal{B}(\mathcal{H})$ the Banach space of bounded linear operators in $\mathcal{H}$ and by $\|\cdot\|$ the associated norm. We recall that the Hilbert space $\mathcal{B}_2(\mathcal{H})$ of Hilbert–Schmidt operators in $\mathcal{H}$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ [35]; the associated scalar product and norm will be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{B}_2}$ and $\|\cdot\|_{\mathcal{B}_2}$, respectively. Another two-sided ideal in $\mathcal{B}(\mathcal{H})$ is the Banach space of trace class operators $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_2(\mathcal{H})$. Given a measure space (X, μ), the locution 'for μ-almost all x in X' will be usually substituted by the symbol $\forall_\mu x \in X$. The following well-known result will turn out to be very useful in section 5. Let the measure space (X, μ) be complete, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}^2(X, \mu; \mathbb{C})$ converging (in norm) to f. If there is a function $\tilde{f} : X \to \mathbb{C}$ such that $\lim_{n\to\infty} f_n(x) = \tilde{f}(x)$, $\forall_\mu x \in X$, then $\tilde{f}$ is μ-measurable and we have that $f = \tilde{f}$, the two functions being regarded as elements of $\mathrm{L}^2(X, \mu; \mathbb{C})$ (i.e. the two functions coincide μ-almost everywhere).

3. Weyl–Wigner quantization–dequantization maps

Every square integrable representation of a l.c.s.c. group G gives rise to an isometry that maps the space of Hilbert–Schmidt operators—acting in the Hilbert space of the representation—into $\mathrm{L}^2(G) \equiv \mathrm{L}^2(G, \mu_G; \mathbb{C})$. Since it transforms operators into functions, it is called the *Wigner (dequantization) map*. Its adjoint, which transforms functions into operators, is called the *Weyl (quantization) map*.

Indeed—see [23–25]—a square integrable projective representation $U : G \to \mathcal{U}(\mathcal{H})$ (with multiplier $\mathtt{m}$) allows us to associate with every Hilbert–Schmidt operator $\hat{A} \in \mathcal{B}_2(\mathcal{H})$ a function $G \ni g \mapsto (\mathfrak{S}_U\hat{A})(g) \in \mathbb{C}$ contained in $\mathrm{L}^2(G)$, in such a way to define a linear map $\mathfrak{S}_U : \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(G)$. To this aim, we exploit the fact that the *finite rank operators* form a dense linear span $\mathsf{FR}(\mathcal{H})$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$. Precisely—denoted by $\hat{D}_U$, as in section 2, the Duflo–Moore operator associated with U (normalized according to μ_G)—consider those operators in $\mathcal{H}$ of the type

$$\widehat{\phi\psi} \equiv |\phi\rangle\langle\psi|, \qquad \phi \in \mathcal{H},\ \psi \in \mathrm{Dom}\big(\hat{D}_U^{-1}\big). \tag{3.1}$$

The linear span generated by the rank one operators of this form—namely,

$$\mathsf{FR}^{\langle|}(\mathcal{H}; U) := \big\{\hat{F} \in \mathsf{FR}(\mathcal{H}) : \ \mathrm{Ran}(\hat{F}^*) = \mathrm{Ker}(\hat{F})^{\perp} \subset \mathrm{Dom}\big(\hat{D}_U^{-1}\big)\big\} \tag{3.2}$$

—is dense in $\mathsf{FR}(\mathcal{H})$ and, hence, in $\mathcal{B}_2(\mathcal{H})$, i.e. $\overline{\mathsf{FR}^{\langle|}(\mathcal{H}; U)} = \mathcal{B}_2(\mathcal{H})$. Explicitly, the elements of $\mathsf{FR}^{\langle|}(\mathcal{H}; U)$ are those operators in $\mathsf{FR}(\mathcal{H})$ that admit a decomposition of the form

$$\hat{F} = \sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle\psi_k|, \qquad \mathsf{N} \in \mathbb{N}, \tag{3.3}$$

where $\{\phi_k\}_{k=1}^{\mathsf{N}}$, $\{\psi_k\}_{k=1}^{\mathsf{N}}$ are linearly independent systems in $\mathcal{H}$, with $\{\psi_k\}_{k=1}^{\mathsf{N}} \subset \mathrm{Dom}(\hat{D}_U^{-1})$. Incidentally, we also introduce another dense linear span in $\mathcal{B}_2(\mathcal{H})$ that will turn out to be useful later on, i.e.

$$\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U) := \{\hat{F} \in \mathsf{FR}(\mathcal{H}) : \mathrm{Ran}(\hat{F}), \mathrm{Ran}(\hat{F}^*) \subset \mathrm{Dom}(\hat{D}_U^{-1})\}. \tag{3.4}$$

At this point, we first define the map $\mathfrak{S}_U$ on all rank one operators of the form (3.1) by setting

$$(\mathfrak{S}_U\, \widehat{\phi\psi})(g) := \mathrm{tr}\,(U(g)^* |\phi\rangle\langle \hat{D}_U^{-1}\psi|) = \langle U(g)\, \hat{D}_U^{-1}\psi, \phi\rangle, \qquad \forall\, \widehat{\phi\psi} \in \mathsf{FR}^{\langle|}(\mathcal{H};U). \tag{3.5}$$

Then, by virtue of the orthogonality relations (2.5), for any $\widehat{\phi_1\psi_1} \equiv |\phi_1\rangle\langle\psi_1|, \widehat{\phi_2\psi_2} \in \mathsf{FR}^{\langle|}(\mathcal{H};U)$, we have

$$\int_G (\mathfrak{S}_U\, \widehat{\phi_1\psi_1})(g)^*\, (\mathfrak{S}_U\, \widehat{\phi_2\psi_2})(g)\, \mathrm{d}\mu_G(g) = \langle\phi_1,\phi_2\rangle\, \langle\psi_2,\psi_1\rangle = \langle \widehat{\phi_1\psi_1}, \widehat{\phi_2\psi_2}\rangle_{\mathcal{B}_2}. \tag{3.6}$$

Thus, extending the definition of the map $\mathfrak{S}_U$ to all $\mathsf{FR}^{\langle|}(\mathcal{H};U)$ by linearity, and next to the whole Hilbert space $\mathcal{B}_2(\mathcal{H})$ by continuity, we obtain an *isometry*—$\mathfrak{S}_U : \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(G)$—i.e. the *(generalized) Wigner map*, or *Wigner transform*, generated by U. It turns out that the range of $\mathfrak{S}_U$, which will be denoted by $\mathcal{R}_U$, depends only on the unitary equivalence class of U. Moreover, as the reader may prove, if the group G is unimodular (hence, $\hat{D}_U = d_U I, d_U > 0$), then for every trace class operator $\hat{\rho} \in \mathcal{B}_1(\mathcal{H})$, we have $(\mathfrak{S}_U\hat{\rho})(g) = d_U^{-1}\ \mathrm{tr}(U(g)^*\hat{\rho})$.

Remark 3.1. Suppose that U is, in particular, a standard unitary representation, and let V be another square integrable unitary representation of G (acting in a Hilbert space $\mathcal{H}'$), unitarily *inequivalent* to U. Then, it is easy to show that

$$(\mathcal{R}_U \equiv \mathrm{Ran}(\mathfrak{S}_U)) \perp \mathrm{Ran}(\mathfrak{S}_V), \tag{3.7}$$

where $\mathfrak{S}_V : \mathcal{B}_2(\mathcal{H}') \to \mathrm{L}^2(G)$ is the Wigner map generated by V.

Remark 3.2. Suppose that the group G is compact—hence, unimodular—and U is a (irreducible) *unitary* representation. Then, by relation (2.13), we have

$$\mathrm{L}^2(G)_{[U]} = \bigoplus_{n=1}^{\delta(U)} \mathrm{Ran}(\mathfrak{W}_U^{\chi_n}) = \mathrm{span}\{\mathsf{c}_{\psi,\phi}^U : \psi,\phi \in \mathcal{H}\} = \mathcal{R}_U, \tag{3.8}$$

where the function $\mathsf{c}_{\psi,\phi}^U \in \mathrm{L}^2(G)$ is the coefficient defined by (2.4). Therefore, by relation (2.12),

$$\mathrm{L}^2(G) = \bigoplus_{U\in\check{G}} \mathcal{R}_U, \tag{3.9}$$

for any realization $\check{G}$ of the unitary dual of G.

We will now explore the 'intertwining properties' of the Wigner map $\mathfrak{S}_U$ with respect to the natural action of the group G in the Hilbert–Schmidt space $\mathcal{B}_2(\mathcal{H})$, and to the standard complex conjugation in $\mathcal{B}_2(\mathcal{H})$.

To this aim, consider the map $U \vee U : G \to \mathcal{U}(\mathcal{B}_2(\mathcal{H}))$ defined by

$$U \vee U(g)\hat{A} := U(g)\,\hat{A}\,U(g)^*, \qquad g \in G, \qquad \hat{A} \in \mathcal{B}_2(\mathcal{H}). \tag{3.10}$$

The map $U \vee U$ is a (strongly continuous) *unitary* representation, even if, in general, the representation U has been assumed to be *projective*. It can be regarded as the standard action of the 'symmetry group' G on the 'quantum-mechanical operators' ('observables' or 'states'). Next, let us consider the map $\mathcal{T}_{\mathtt{m}} : G \to \mathcal{U}(\mathrm{L}^2(G))$ defined by

$$(\mathcal{T}_{\mathtt{m}}(g)f)(g') := \Delta_G(g)^{\frac{1}{2}}\, \overset{\leftrightarrow}{\mathtt{m}}(g,g')\, f(g^{-1}g'g), \qquad g,g' \in G, \qquad f \in \mathrm{L}^2(G), \tag{3.11}$$

where the function $\overset{\leftrightarrow}{\mathtt{m}} : G \times G \to \mathbb{T}$ has the following expression:

$$\overset{\leftrightarrow}{\mathtt{m}}(g, g') := \mathtt{m}(g, g^{-1}g')^* \mathtt{m}(g^{-1}g', g). \tag{3.12}$$

As the reader may check, also the map $\mathcal{T}_{\mathtt{m}}$ is a unitary representation. For $\mathtt{m} \equiv 1$, it coincides with the restriction to the 'diagonal subgroup' of the *two-sided regular representation* of the direct product group $G \times G$; see [27, 36]. The link between the unitary representations defined by (3.10) and (3.11) is provided by the following result.

Proposition 3.1. *The Wigner transform $\mathfrak{S}_U$ intertwines the representation $U \vee U$ with the representation $\mathcal{T}_{\mathtt{m}}$, namely*

$$\mathfrak{S}_U U \vee U(g) = \mathcal{T}_{\mathtt{m}}(g)\, \mathfrak{S}_U, \qquad \forall g \in G. \tag{3.13}$$

Therefore, $\mathcal{R}_U$ is an invariant subspace for the unitary representation $\mathcal{T}_{\mathtt{m}}$ and the representation $U \vee U$ is unitarily equivalent to a subrepresentation of $\mathcal{T}_{\mathtt{m}}$, i.e. to the restriction of $\mathcal{T}_{\mathtt{m}}$ to $\mathcal{R}_U$.

Proof. One can easily prove that $\mathfrak{S}_U U \vee U(g)\widehat{\phi\psi} = \mathcal{T}_{\mathtt{m}}(g)\, \mathfrak{S}_U\, \widehat{\phi\psi}$, for any rank one operator $\widehat{\phi\psi}$ of the form (3.1). This relation extends to the linear span generated by the rank one operators of such form, i.e. to the dense linear span $\mathsf{FR}^{\langle|}(\mathcal{H}; U)$. Therefore, the bounded operators $\mathfrak{S}_U U \vee U(g)$ and $\mathcal{T}_{\mathtt{m}}(g)\, \mathfrak{S}_U$ coincide on a dense linear span in $\mathcal{B}_2(\mathcal{H})$; hence, they are equal. □

Let us consider, now, the *antilinear* map $\mathsf{J}_{\mathtt{m}} : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ defined by

$$(\mathsf{J}_{\mathtt{m}} f)(g) := \Delta_G(g)^{-\frac{1}{2}}\, \mathtt{m}(g, g^{-1})\, f(g^{-1})^*, \qquad \forall f \in \mathrm{L}^2(G). \tag{3.14}$$

We leave to the reader the easy task of verifying that the map $\mathsf{J}_{\mathtt{m}}$ is (well defined and) a *complex conjugation* in $\mathrm{L}^2(G)$: $\mathsf{J}_{\mathtt{m}} = \mathsf{J}_{\mathtt{m}}^*$ and $\mathsf{J}_{\mathtt{m}}^2 = I$ (i.e. $\mathsf{J}_{\mathtt{m}}$ is a self-adjoint antiunitary map).

Proposition 3.2. *The isometry $\mathfrak{S}_U$ intertwines the standard complex conjugation*

$$\mathfrak{J} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}^* \in \mathcal{B}_2(\mathcal{H}) \tag{3.15}$$

in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ with the complex conjugation $\mathsf{J}_{\mathtt{m}}$ in $\mathrm{L}^2(G)$, namely

$$\mathfrak{S}_U \mathfrak{J} = \mathsf{J}_{\mathtt{m}} \mathfrak{S}_U. \tag{3.16}$$

Therefore, $\mathcal{R}_U$ is an invariant subspace for the complex conjugation $\mathsf{J}_{\mathtt{m}}$.

Proof. The proof is analogous to the proof of proposition 3.1; we leave the details to the reader. Hint: this time prove that relation (3.16) holds in the dense linear span $\mathsf{FR}^{|\rangle\langle|}(\mathcal{H}; U)$, at first. □

Since the generalized Wigner transform $\mathfrak{S}_U$ is an isometry, its adjoint $\mathfrak{S}_U^* : \mathrm{L}^2(G) \to \mathcal{B}_2(\mathcal{H})$ is a partial isometry such that $\mathfrak{S}_U^*\, \mathfrak{S}_U = I$ and $\mathfrak{S}_U\, \mathfrak{S}_U^* = \mathrm{P}_{\mathcal{R}_U}$, where $\mathrm{P}_{\mathcal{R}_U}$ is the orthogonal projection onto the (closed) subspace $\mathcal{R}_U \equiv \mathrm{Ran}(\mathfrak{S}_U) = \mathrm{Ker}(\mathfrak{S}_U^*)$ of $\mathrm{L}^2(G)$. Thus, the partial isometry $\mathfrak{S}_U^*$ is the pseudo-inverse of $\mathfrak{S}_U$, and we will call it *(generalized) Weyl map* generated by the representation U.

Let us provide an expression of the Weyl map. As is well known, the weak integral

$$\hat{U}(\mathsf{f}) := \int_G \mathsf{f}(g)\, U(g)\, \mathrm{d}\mu_G(g), \qquad \forall \mathsf{f} \in \mathrm{L}^1(G), \tag{3.17}$$

defines a bounded operator in $\mathcal{H}$ (here the square integrability of U does not play any role). Then, one can easily prove the following result.

Proposition 3.3. *For every* $\mathsf{f} \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$*, the densely defined operator* $\hat{U}(\mathsf{f})\hat{D}_U^{-1}$ *extends to a Hilbert–Schmidt operator and*

$$\overline{\hat{U}(\mathsf{f})\hat{D}_U^{-1}} = \mathfrak{S}_U^* \mathsf{f}. \tag{3.18}$$

Therefore, for every function $f \in \mathrm{L}^2(G)$*—given a sequence* $\{\mathsf{f}_n\}_{n\in\mathbb{N}}$ *in* $\mathrm{L}^2(G)$*, contained in the dense linear span* $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$*, such that* $\|\cdot\|_{\mathrm{L}^2} \lim_{n\to\infty} \mathsf{f}_n = f$*—we have*

$$\mathfrak{S}_U^* f = \|\cdot\|_{\mathcal{B}_2} \lim_{n\to\infty} \mathfrak{S}_U^* \mathsf{f}_n = \|\cdot\|_{\mathcal{B}_2} \lim_{n\to\infty} \overline{\hat{U}(\mathsf{f}_n)\hat{D}_U^{-1}}. \tag{3.19}$$

In the case where the group G is unimodular, the following weak integral formula holds:

$$\mathfrak{S}_U^* f = d_U^{-1} \int_G f(g)\, U(g)\, \mathrm{d}\mu_G(g), \qquad \forall f \in \mathrm{L}^2(G). \tag{3.20}$$

We will finally establish a result that will be useful in section 6. We leave the (straightforward) proof of this result to the reader.

Proposition 3.4. *Suppose that the Hilbert space* $\mathcal{H}$ *of the representation U is a space* $\mathrm{L}^2(X) \equiv \mathrm{L}^2(X, \mu; \mathbb{C})$ *of square integrable functions on a σ-finite measure space* (X, μ)*. Then, for every* $\mathsf{f} \in \mathrm{L}^1(G)$ *and every* $\phi \in \mathrm{L}^2(X)$*, the function* $G \ni g \mapsto \mathsf{f}(g)\,(U(g)\,\phi)(x) \in \mathbb{C}$ *belongs to* $\mathrm{L}^1(G)$ *for μ-a.a.* $x \in X$*, and the following relation holds:*

$$(\hat{U}(\mathsf{f})\phi)(x) = \int_G \mathsf{f}(g)\,(U(g)\,\phi)(x)\, \mathrm{d}\mu_G(g), \qquad \forall_\mu x \in X. \tag{3.21}$$

Therefore, for every $\mathsf{f} \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ *and every* $\varphi \in \mathrm{Dom}\big(\hat{D}_U^{-1}\big) \subset \mathrm{L}^2(X)$*, we have*

$$((\mathfrak{S}_U^* \mathsf{f})\varphi)(x) = \int_G \mathsf{f}(g)\,\big(U(g)\,\hat{D}_U^{-1}\varphi\big)(x)\, \mathrm{d}\mu_G(g), \qquad \forall_\mu x \in X. \tag{3.22}$$

4. Star products from quantization–dequantization maps

In this section, we will show that the quantization–dequantization maps previously introduced induce, in a natural way, a 'star product of functions' enjoying remarkable properties. Let U be a square integrable (irreducible) projective representation of the l.c.s.c. group G in the Hilbert space $\mathcal{H}$, and let $\mathfrak{S}_U : \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(G)$ be the associated Wigner map. Consider the following bilinear map from $\mathrm{L}^2(G) \times \mathrm{L}^2(G)$ into $\mathrm{L}^2(G)$:

$$(\cdot) \overset{U}{\star} (\cdot) : \mathrm{L}^2(G) \times \mathrm{L}^2(G) \ni (f_1, f_2) \mapsto \mathfrak{S}_U\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big) \in \mathrm{L}^2(G), \tag{4.1}$$

i.e. $f_1 \overset{U}{\star} f_2$ is the function obtained dequantizing the product (composition) of the two operators which are the 'quantized versions' of the functions f_1, f_2. We will call the bilinear map (4.1) the *star product associated with the representation U*.

Before considering the properties of the star product associated with U, it is worth fixing some terminology about algebras. By a *Banach algebra,* we mean an associative algebra $\mathcal{A}$ which is a Banach space (with norm $\|\cdot\|_{\mathcal{A}}$) such that

$$\|ab\|_{\mathcal{A}} \leqslant \|a\|_{\mathcal{A}}\, \|b\|_{\mathcal{A}}, \qquad \forall a, b \in \mathcal{A}. \tag{4.2}$$

Given Banach algebras $\mathcal{A}$ and $\mathcal{A}'$, we will say that a linear map $\mathfrak{E} : \mathcal{A} \to \mathcal{A}'$ is an (isometric) *isomorphism of Banach algebras* if it is a surjective isometry such that $\mathfrak{E}(ab) = \mathfrak{E}(a)\mathfrak{E}(b)$, for all $a, b \in \mathcal{A}$.

A Banach algebra $\mathcal{A}$—endowed with an involution[2] ($a \mapsto a^*$)—such that

$$\|a\|_{\mathcal{A}} = \|a^*\|_{\mathcal{A}}, \qquad \forall\, a \in \mathcal{A} \tag{4.3}$$

will be called a *Banach* $*$*-algebra* (Banach star-algebra; of course, the 'star' in $*$-algebra, which refers to an involution, should not generate confusion with the 'star' product).

A Banach $*$-algebra $\mathcal{A}$ is said to be a H**-algebra* [37, 38] if, in addition, it is a (separable complex) Hilbert space (with $\|a\|_{\mathcal{A}} = \sqrt{\langle a, a\rangle_{\mathcal{A}}}$) satisfying

$$\langle ab, c\rangle_{\mathcal{A}} = \langle b, a^* c\rangle_{\mathcal{A}} \qquad \text{and} \qquad \langle ab, c\rangle_{\mathcal{A}} = \langle a, cb^*\rangle_{\mathcal{A}}, \qquad \forall\, a, b, c \in \mathcal{A}. \tag{4.4}$$

Clearly, condition (4.3) now means that the involution $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ is a complex conjugation (an idempotent antiunitary operator). For every element x of a $\mathcal{A}$, the two relations $x\mathcal{A} = \{0\}$ and $\mathcal{A}x = \{0\}$ turn out to be equivalent. The *annihilator ideal* of $\mathcal{A}$ is the set $\mathcal{A}_0$ defined by

$$\mathcal{A}_0 := \{x \in \mathcal{A} : x\mathcal{A} = \{0\}\} = \{x \in \mathcal{A} : \mathcal{A}x = \{0\}\}. \tag{4.5}$$

The annihilator ideal is a self-adjoint (i.e. for every $x \in \mathcal{A}_0$, x^* belongs to $\mathcal{A}_0$ as well) closed two-sided ideal in $\mathcal{A}$. The H*-algebra $\mathcal{A}$ is said to be *proper* (or *semi-simple*) if it satisfies the following two equivalent conditions:

$$(x \in \mathcal{A},\ \ x\mathcal{A} = \{0\} \ \ \Rightarrow\ \ x = 0) \qquad \text{and} \qquad (x \in \mathcal{A},\ \ \mathcal{A}x = \{0\} \ \ \Rightarrow\ \ x = 0)\,, \tag{4.6}$$

namely, if $\mathcal{A}_0 = \{0\}$. Every H*-algebra $\mathcal{A}$ admits an orthogonal sum decomposition of the following type:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1, \tag{4.7}$$

where $\mathcal{A}_0$ is the annihilator ideal of $\mathcal{A}$, and $\mathcal{A}_1$ is a self-adjoint closed two-sided ideal which (endowed with the restrictions of the algebra operation and of the involution of $\mathcal{A}$) is a proper H*-algebra. We will call $\mathcal{A}_1$ the *canonical ideal* of $\mathcal{A}$, and we will denote by $\mathrm{P}_{\mathcal{A}_1}$ the orthogonal projection onto $\mathcal{A}_1$. The canonical ideal is characterized by the relation

$$ab = (\mathrm{P}_{\mathcal{A}_1} a)(\mathrm{P}_{\mathcal{A}_1} b), \qquad \forall\, a, b \in \mathcal{A}, \tag{4.8}$$

in the following sense. Suppose that $\tilde{\mathcal{A}} \subset \mathcal{A}$ is a closed two-sided ideal, which is a proper H*-algebra such that $ab = (\mathrm{P}_{\tilde{\mathcal{A}}} a)(\mathrm{P}_{\tilde{\mathcal{A}}} b)$, $\forall\, a, b \in \mathcal{A}$. Then, it is easy to show that $\tilde{\mathcal{A}} = \mathcal{A}_1$.

A linear map $\mathfrak{C} : \mathcal{A} \to \mathcal{A}'$—where $\mathcal{A}, \mathcal{A}'$ are H*-algebras—is said to be an *isomorphism of* H**-algebras* if it is a unitary operator such that

$$\mathfrak{C}(ab) = \mathfrak{C}(a)\mathfrak{C}(b) \qquad \text{and} \qquad \mathfrak{C}(a^*) = \mathfrak{C}(a)^*, \qquad \forall\, a, b \in \mathcal{A}. \tag{4.9}$$

As is well known, the Hilbert space $\mathcal{B}_2(\mathcal{H})$ is a proper H*-algebra with respect to the ordinary composition of operators (algebra operation) and to the standard complex conjugation $\mathfrak{J}$ (involution), see (3.15).

The star product defined above is characterized by the following result, whose proof, being straightforward, is left to the reader.

Proposition 4.1. *The bilinear map* $(\cdot) \overset{U}{\star} (\cdot) : \mathrm{L}^2(G) \times \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ *associated with the square integrable projective representation U enjoys the following properties:*

(i) the vector space $\mathrm{L}^2(G)$*, endowed with the operation* $(\cdot) \overset{U}{\star} (\cdot)$*, is an associative algebra;*

[2] Let V be a vector space, and $(\cdot,\cdot) : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ a bilinear operation in V. We recall that an an *involution* in V, with respect to the bilinear operation $(\cdot,\cdot)$, is an antilinear map $\mathsf{V} \ni a \mapsto a^* \in \mathsf{V}$ satisfying $(a^*)^* = a$ and $(a, b)^* = (b^*, a^*)$, $\forall\, a, b \in \mathsf{V}$.

(ii) *the antilinear map* $\mathsf{J}_{\mathtt{m}}$ *is an involution in the vector space* $\mathrm{L}^2(G)$ *with respect to the bilinear operation* $(\cdot)\overset{U}{\star}(\cdot)$*, i.e.*

$$\mathsf{J}_{\mathtt{m}}(\mathsf{J}_{\mathtt{m}}f) = f \qquad \text{and} \qquad \mathsf{J}_{\mathtt{m}}(f_1 \overset{U}{\star} f_2) = (\mathsf{J}_{\mathtt{m}}f_2)\overset{U}{\star}(\mathsf{J}_{\mathtt{m}}f_1), \qquad \forall\, f, f_1, f_2 \in \mathrm{L}^2(G); \tag{4.10}$$

(iii) $\mathrm{L}^2(G)$*—regarded as a Banach space with respect to the norm* $\|\cdot\|_{\mathrm{L}^2}$*, and endowed with the the star product associated with U and with the involution* $\mathsf{J}_{\mathtt{m}}$*—is a Banach* $*$*-algebra; in particular, it satisfies the relation*

$$\big\| f_1 \overset{U}{\star} f_2\big\|_{\mathrm{L}^2} \leqslant \|f_1\|_{\mathrm{L}^2}\|f_2\|_{\mathrm{L}^2}, \qquad \forall\, f_1, f_2 \in \mathrm{L}^2(G); \tag{4.11}$$

(iv) $\mathcal{A}_U \equiv (\mathrm{L}^2(G), (\cdot)\overset{U}{\star}(\cdot), \mathsf{J}_{\mathtt{m}})$ *is a* H^**-algebra; indeed, for all* $f_1, f_2, f_3 \in \mathrm{L}^2(G)$*,*

$$\big\langle f_1 \overset{U}{\star} f_2, f_3\big\rangle_{\mathrm{L}^2} = \big\langle f_2, (\mathsf{J}_{\mathtt{m}}f_1)\overset{U}{\star} f_3\big\rangle_{\mathrm{L}^2} \qquad \text{and}$$
$$\big\langle f_1 \overset{U}{\star} f_2, f_3\big\rangle_{\mathrm{L}^2} = \big\langle f_1, f_3 \overset{U}{\star} (\mathsf{J}_{\mathtt{m}}f_2)\big\rangle_{\mathrm{L}^2}; \tag{4.12}$$

(v) *for any* $f_1, f_2 \in \mathrm{L}^2(G)$*, we have that*

$$f_1 \overset{U}{\star} f_2 \in \mathcal{R}_U; \tag{4.13}$$

therefore, the (closed) subspace $\mathcal{R}_U \equiv \mathrm{Ran}(\mathfrak{S}_U)$ *of* $\mathrm{L}^2(G)$ *is a closed two-sided ideal in* $\mathcal{A}_U$ *and—endowed with the restrictions of the star product associated with U and of the involution* $\mathsf{J}_{\mathtt{m}}$ *(*$\mathcal{R}_U$ *is an invariant subspace for* $\mathsf{J}_{\mathtt{m}}$*, see proposition 3.2)—is a* H^**-algebra;*

(vi) *the* H^**-algebra* $\mathcal{R}_U$ *is proper and, for any* $f_1, f_2 \in \mathrm{L}^2(G)$*, we have that*

$$f_1 \overset{U}{\star} f_2 = \big(\mathrm{P}_{\mathcal{R}_U} f_1\big)\overset{U}{\star}\big(\mathrm{P}_{\mathcal{R}_U} f_2\big); \tag{4.14}$$

hence, $\mathcal{R}_U$ *and its orthogonal complement* $\mathcal{R}_U^{\perp}$ *are, respectively, the canonical ideal and the annihilator ideal of* $\mathcal{A}_U$*, and the* H^**-algebra* $\mathcal{A}_U$ *is proper if and only if* $\mathcal{R}_U = \mathrm{L}^2(G)$*;*

(vii) *the unitary operator*

$$\mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \mathfrak{S}_U\hat{A} \in \mathcal{R}_U \tag{4.15}$$

is an isomorphism of (proper) H^**-algebras;*

(viii) *the canonical ideal* $\mathcal{R}_U$ *is an invariant subspace for the representation* $\mathcal{T}_{\mathtt{m}}$ *—(see (3.11))— and the star product associated with U is equivariant with respect to this representation, i.e.*

$$\mathcal{T}_{\mathtt{m}}(g)\big(f_1 \overset{U}{\star} f_2\big) = (\mathcal{T}_{\mathtt{m}}(g)f_1)\overset{U}{\star}(\mathcal{T}_{\mathtt{m}}(g)f_2), \qquad \forall\, f_1, f_2 \in \mathrm{L}^2(G),\ \forall\, g \in G. \tag{4.16}$$

It is interesting to note that the definition of the star product (4.1) can be suitably generalized. In fact, since $\mathcal{B}_2(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, with every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$ is associated a bilinear map $(\cdot)\underset{\hat{K}}{\circ}(\cdot) : \mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H}) \to \mathcal{B}_2(\mathcal{H})$—the $\hat{K}$*-product* (this notion has been considered for 'generic operators' in [19, 20]) in $\mathcal{B}_2(\mathcal{H})$—defined by

$$\hat{A}\underset{\hat{K}}{\circ}\hat{B} := \hat{A}\hat{K}\hat{B}, \qquad \forall\, \hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H}). \tag{4.17}$$

Observe that $\mathcal{B}_2(\mathcal{H})$, endowed with the operation $(\cdot)\underset{\hat{K}}{\circ}(\cdot)$, is an associative algebra, and, if $\hat{K}$ is self-adjoint, then $\mathfrak{J}$ is an involution in $\mathcal{B}_2(\mathcal{H})$ with respect to this operation. Moreover—since $\|\hat{A}\hat{K}\hat{B}\|_{\mathcal{B}_2} \leqslant \|\hat{K}\|\|\hat{A}\|_{\mathcal{B}_2}\|\hat{B}\|_{\mathcal{B}_2}$—it is clear that if $\|\hat{K}\| \leqslant 1$, then $(\mathcal{B}_2(\mathcal{H}), (\cdot)\underset{\hat{K}}{\circ}(\cdot))$ is a Banach algebra; if, furthermore, $\hat{K}$ is self-adjoint, then $(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot), \mathfrak{J})$ is a Banach $*$-algebra. The operation (4.17) allows us to introduce the following bilinear map:

$$(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot) : \mathrm{L}^2(G) \times \mathrm{L}^2(G) \ni (f_1, f_2) \mapsto \mathfrak{S}_U\big((\mathfrak{S}_U^* f_1) \underset{\hat{K}}{\circ} (\mathfrak{S}_U^* f_2)\big) \in \mathrm{L}^2(G). \tag{4.18}$$

We will call the operation (4.18) *$\hat{K}$-deformed star product associated with U*. Obviously, the $\hat{K}$-deformed star product coincides with the star product defined by (4.1) in the case where $\hat{K} = I$. The main properties of $\hat{K}$-deformed star product are summarized by the following proposition, whose proof we leave to the reader.

Proposition 4.2. *For every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$, the bilinear map $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$: $\mathrm{L}^2(G) \times \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ enjoys the following properties:*

(i) the vector space $\mathrm{L}^2(G)$, endowed with the operation $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$, is an associative algebra;

(ii) in the case where the operator $\hat{K}$ is self-adjoint, the antilinear map $\mathsf{J}_{\mathtt{m}}$ is an involution in the vector space $\mathrm{L}^2(G)$ with respect to the bilinear operation $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$, i.e.

$$\mathsf{J}_{\mathtt{m}}(\mathsf{J}_{\mathtt{m}} f) = f \qquad \textit{and} \qquad \mathsf{J}_{\mathtt{m}}\big(f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2\big) = (\mathsf{J}_{\mathtt{m}} f_2) \overset{U}{\underset{\hat{K}}{\star}} (\mathsf{J}_{\mathtt{m}} f_1), \quad \forall\, f, f_1, f_2 \in \mathrm{L}^2(G); \tag{4.19}$$

(iii) if $\|\hat{K}\| \leqslant 1$, then $\mathrm{L}^2(G)$—regarded as a Banach space with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$, and endowed with the $\hat{K}$-deformed star product associated with U—is a Banach algebra; in particular, it satisfies the relation

$$\big\| f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 \big\|_{\mathrm{L}^2} \leqslant \|f_1\|_{\mathrm{L}^2} \|f_2\|_{\mathrm{L}^2}, \qquad \forall\, f_1, f_2 \in \mathrm{L}^2(G); \tag{4.20}$$

if, furthermore, the operator $\hat{K}$ is self-adjoint, then $\big(\mathrm{L}^2(G), (\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot), \mathsf{J}_{\mathtt{m}}\big)$ is a Banach $*$-algebra;

(iv) for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have that

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 \in \mathcal{R}_U; \tag{4.21}$$

therefore—assuming that $\|\hat{K}\| \leqslant 1$—the (closed) subspace $\mathcal{R}_U$ of $\mathrm{L}^2(G)$ is a closed two-sided ideal in the Banach algebra $(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot))$;

(v) for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have that

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 = \big(\mathrm{P}_{\mathcal{R}_U} f_1\big) \overset{U}{\underset{\hat{K}}{\star}} \big(\mathrm{P}_{\mathcal{R}_U} f_2\big); \tag{4.22}$$

(vi) assuming that $\|\hat{K}\| \leqslant 1$, the application

$$\mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \mathfrak{S}_U \hat{A} \in \mathcal{R}_U \tag{4.23}$$

is an isomorphism of the Banach algebras $(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot))$ and $\big(\mathcal{R}_U, (\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)\big)$.

5. Main results: explicit formulas for star products

The aim of this section is to provide suitable expressions for the star products associated with square integrable representations that have been defined and characterized in section 4. For the sake of clarity, we will split our presentation into a few subsections.

5.1. Assumptions and further notations

In the following, we will always assume that U is a square integrable (irreducible) projective representation—with multiplier $\mathtt{m}$—of the l.c.s.c. group G in the Hilbert space $\mathcal{H}$. We will denote, as usual, by $\hat{D}_U$ the associated Duflo–Moore operator, normalized according to a given left Haar measure μ_G on G. Recall that, if G is unimodular, then $\hat{D}_U = d_U I, d_U > 0$; otherwise, $\hat{D}_U$ is unbounded. We will use—often without any further explanation—the notations and the tools introduced in sections 2–4; in particular, we will exploit the orthogonality relations for square integrable representations and the result recalled at the end of section 2.

Before starting our program, it is worth fixing a few additional notations. It will be convenient to adopt the shorthand notation $\int \mathrm{d}\mu_G$ for the integral $\int_G \mathrm{d}\mu_G$. We will denote by ${}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{n\to\infty}$ the limit of a sequence in $\mathrm{L}^2(G)$ (converging with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$). Given a finite or countably infinite index set $\mathcal{N} = \{n\}$, we denote by ${}_{\|\cdot\|_{\mathrm{L}^2}}\sum_n$ either simply a finite sum in $\mathrm{L}^2(G)$ ($\mathcal{N}$ finite), or an infinite sum in $\mathrm{L}^2(G)$ converging with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$. Clearly, an analogous meaning will be understood for the symbol ${}_{\|\cdot\|_{\mathcal{B}_2}}\sum_n$ (of course, in this case the relevant space is $\mathcal{B}_2(\mathcal{H})$), or, in general, ${}_{\|\cdot\|}\sum_n$. Given a bounded operator $\hat{B}$ in $\mathcal{H}$, we can define two natural bounded operators in the Hilbert–Schmidt space $\mathcal{B}_2(\mathcal{H})$, i.e. the operators

$$\mathfrak{L}_{\hat{B}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{B}\hat{A} \in \mathcal{B}_2(\mathcal{H}), \qquad \mathfrak{R}_{\hat{B}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}\,\hat{B} \in \mathcal{B}_2(\mathcal{H}). \tag{5.1}$$

It is obvious that $\mathfrak{L}_{\hat{B}}\mathfrak{R}_{\hat{B}'} = \mathfrak{R}_{\hat{B}'}\mathfrak{L}_{\hat{B}}$. In particular, given a vector $\chi \in \mathcal{H}$, we will denote by $\mathfrak{R}_{\widehat{\chi}}$ the bounded linear operator in $\mathcal{B}_2(\mathcal{H})$ defined by

$$\mathfrak{R}_{\widehat{\chi}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}\,\widehat{\chi} \in \mathcal{B}_2(\mathcal{H}), \tag{5.2}$$

where we set: $\widehat{\chi} \equiv \widehat{\chi\chi} \equiv |\chi\rangle\langle\chi|$. It is clear that—for χ nonzero and normalized—$\mathfrak{R}_{\widehat{\chi}}$ is an orthogonal projector in the Hilbert space $\mathcal{B}_2(\mathcal{H})$.

Remark 5.1. Let J be a complex conjugation in $\mathcal{H}$ (a self-adjoint antiunitary operator). Then, the bounded linear map $\mathfrak{U}_J : \mathcal{H}\otimes\mathcal{H} \to \mathcal{B}_2(\mathcal{H})$, determined (in a consistent way) by

$$\mathfrak{U}_J\phi\otimes\psi = |\phi\rangle\langle J\psi|, \qquad \forall\,\phi,\psi\in\mathcal{H}, \tag{5.3}$$

is a unitary operator (indeed, it is an isometry on the dense linear span generated by the separable elements of $\mathcal{H}\otimes\mathcal{H}$, and the image of this linear span is $\mathsf{FR}(\mathcal{H})$, which is dense in $\mathcal{B}_2(\mathcal{H})$). It is easy to check that $\mathfrak{U}_J(I\otimes\widehat{\chi})\mathfrak{U}_J^* = \mathfrak{R}_{\widehat{\chi}'}$, where $\widehat{\chi}' = J\widehat{\chi}J = |J\chi\rangle\langle J\chi|$. Let $\{\chi_n\}_{n\in\mathcal{N}}$ be an orthonormal basis in $\mathcal{H}$. One can always choose the complex conjugation J in such a way that $J\chi_n = \chi_n$, for any $n\in\mathcal{N}$; hence: $\mathfrak{U}_J(I\otimes\widehat{\chi}_n)\mathfrak{U}_J^* = \mathfrak{R}_{\widehat{\chi}_n}$, with $\widehat{\chi}_n \equiv |\chi_n\rangle\langle\chi_n|$. This choice of J is convenient for noting the fact that the relation ${}_{\|\cdot\|}\sum_n(I\otimes\widehat{\chi}_n\Psi) = \Psi$, $\forall\,\Psi\in\mathcal{H}\otimes\mathcal{H}$ is equivalent to ${}_{\|\cdot\|_{\mathcal{B}_2}}\sum_n\mathfrak{R}_{\widehat{\chi}_n}\hat{A} = \hat{A}, \forall\,\hat{A}\in\mathcal{B}_2(\mathcal{H})$.

Besides, given a vector χ contained in the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-1}\big)$, let $\check{\chi}$ be the linear operator in $\mathcal{H}$, of rank at most one, defined by

$$\check{\chi} := |\chi\rangle\big\langle\hat{D}_U^{-1}\chi\big|. \tag{5.4}$$

Then, we can consider the bounded linear operator $\mathfrak{R}_{\check{\chi}} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}\,\check{\chi} \in \mathcal{B}_2(\mathcal{H})$. Note that, if the group G is unimodular, we have $\mathfrak{R}_{\check{\chi}} = d_U^{-1}\,\mathfrak{R}_{\widehat{\chi}}$.

Let us also introduce two integral kernels. Our formulas for star products will be based on these kernels. First—for any bounded operator $\hat{K}$ in $\mathcal{H}$ and any vector $\chi\in\mathcal{H}$, contained in the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$—consider the integral kernel $\varkappa_U(\hat{K},\chi;\cdot,\cdot) : G\times G \to \mathbb{C}$ defined by

$$\varkappa_U(\hat{K},\chi;g,h) := \big\langle U(g)\,\hat{D}_U^{-2}\chi,\,\hat{K}\,U(h)\,\hat{D}_U^{-1}\chi\big\rangle = \big\langle\hat{K}^*U(g)\,\hat{D}_U^{-2}\chi,\,U(h)\,\hat{D}_U^{-1}\chi\big\rangle. \tag{5.5}$$

For notational convenience, we set $\varkappa_U(\chi; g, h) \equiv \varkappa_U(I, \chi; g, h) = \langle U(g)\,\hat{D}_U^{-2}\chi, U(h)\,\hat{D}_U^{-1}\chi\rangle$. Next, again for every vector χ contained in $\mathrm{Dom}(\hat{D}_U^{-2})$, let $\kappa_U(\chi;\cdot,\cdot,\cdot) : G \times G \times G \to \mathbb{C}$ be the integral kernel defined by

$$\kappa_U(\chi; g, h, h') := \langle U(g)\,\hat{D}_U^{-1}\chi, U(h)\,\hat{D}_U^{-1}\,U(h')\,\hat{D}_U^{-1}\chi\rangle. \tag{5.6}$$

Exploiting relation (2.8) and the fact that

$$\begin{aligned} U(h^{-1}g) = \mathtt{m}(h^{-1}, g)\,U(h^{-1})\,U(g) &= \mathtt{m}(h^{-1}, g)\,\mathtt{m}(h, h^{-1})^*\,U(h)^*U(g) \\ &= \mathtt{m}(h, h^{-1}g)^*\,U(h)^*U(g), \end{aligned} \tag{5.7}$$

we find

$$\kappa_U(\chi; g, h, h') = \mathtt{m}(h, h^{-1}g)^*\Delta_G(h^{-1}g)^{\frac{1}{2}}\,\varkappa_U(\chi; h^{-1}g, h'), \qquad \forall\, g, h, h' \in G. \tag{5.8}$$

Observe that—since $\varkappa_U(\hat{K}, \chi; g, \cdot) = \mathfrak{S}_U\big(|\hat{K}^*U(g)\,\hat{D}_U^{-2}\chi\rangle\langle\chi|\big)^*$—for any $g \in G$, we have that the function $G \ni h \mapsto \varkappa_U(\hat{K}, \chi; g, h) \in \mathbb{C}$ belongs to $\mathrm{L}^2(G)$. Moreover, by relation (5.8), for any $g, h \in G$, the function $G \ni h' \mapsto \kappa_U(\chi; g, h, h') \in \mathbb{C}$ belongs to $\mathrm{L}^2(G)$, as well.

5.2. Preliminary results

The following result will turn out to be fundamental for our purposes.

Proposition 5.1. *For every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$, for every function $f \in \mathrm{L}^2(G)$ and for every vector $\chi \in \mathrm{Dom}(\hat{D}_U^{-2})$, the following formula holds:*

$$\big(\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{L}_{\hat{K}}\mathfrak{S}_U^* f\big)(g) = \int \mathrm{d}\mu_G(h)\,\varkappa_U(\hat{K}, \chi; g, h)\,f(h), \qquad \forall_{\mu_G} g \in G. \tag{5.9}$$

Proof. Indeed, for every $f \in \mathrm{L}^2(G)$, we have

$$\begin{aligned} \int \mathrm{d}\mu_G(h)\,\varkappa_U(\hat{K}, \chi; g, h)\,f(h) &= \langle \mathfrak{S}_U\big(|\hat{K}^*U(g)\,\hat{D}_U^{-2}\chi\rangle\langle\chi|\big), f\rangle_{\mathrm{L}^2} \\ &= \langle \hat{K}^*|U(g)\,\hat{D}_U^{-2}\chi\rangle\langle\chi|, \mathfrak{S}_U^* f\rangle_{\mathcal{B}_2} \\ &= \langle U(g)\,\hat{D}_U^{-2}\chi, \hat{K}\big(\mathfrak{S}_U^* f\big)\chi\rangle, \qquad \forall_{\mu_G} g \in G. \end{aligned} \tag{5.10}$$

Hence, we conclude that

$$\begin{aligned} \int \mathrm{d}\mu_G(h)\,\varkappa_U(\hat{K}, \chi; g, h)\,f(h) &= \big(\mathfrak{S}_U\big(\hat{K}\big(\mathfrak{S}_U^* f\big)|\chi\rangle\langle\hat{D}_U^{-1}\chi|\big)\big)(g) \\ &= (\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{L}_{\hat{K}}\mathfrak{S}_U^* f)(g), \end{aligned} \tag{5.11}$$

$\forall_{\mu_G} g \in G$. The proof of formula (5.9) is complete. □

At this point, in order to prove the main result of the paper—i.e. theorem 5.1—we need to pass through three technical results. The third one (lemma 5.3) ‘essentially contains’ the expression of the star product, already, but it requires a refinement (see proposition 5.2 below) before getting to the main theorem swiftly.

Lemma 5.1. *For every $f \in \mathrm{L}^2(G)$ and for every $g \in G$, the following relation holds:*

$$(R_{\mathtt{m}}(g)\mathrm{J}_{\mathtt{m}} f)(h)^* = \mathtt{m}(h, h^{-1}g)^*\Delta_G(h^{-1}g)^{\frac{1}{2}}\,f(h^{-1}g), \tag{5.12}$$

$\forall_{\mu_G} h \in G$. Therefore, for any $f_1, f_2 \in \mathrm{L}^2(G)$ and for every $g \in G$, the function

$$G \ni h \mapsto f_1(h)\,\mathtt{m}(h, h^{-1}g)^*\Delta_G(h^{-1}g)^{\frac{1}{2}}\,f_2(h^{-1}g) \in \mathbb{C} \tag{5.13}$$

belongs to $\mathrm{L}^1(G)$ *and*

$$\int \mathrm{d}\mu_G(h)\; f_1(h)\, \mathtt{m}(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}} f_2(h^{-1}g) = \langle R_{\mathtt{m}}(g) \mathrm{J}_{\mathtt{m}} f_2, f_1\rangle_{\mathrm{L}^2}. \tag{5.14}$$

Proof. Use the definition of the representation $R_{\mathtt{m}} : G \to \mathcal{U}(\mathrm{L}^2(G))$ (see (2.10)) and of the complex conjugation $\mathrm{J}_{\mathtt{m}} : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ (see (3.14)), and then suitably exploit formula (2.3) for manipulating multipliers. □

Lemma 5.2. *For any* $f_1, f_2 \in \mathrm{L}^2(G)$ *and for every* $\chi \in \mathrm{Dom}\big(\hat{D}_U^{-2}\big)$*, the following relation holds:*

$$\int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h')\, f_1(h) f_2(h') = \big\langle R_{\mathtt{m}}(g) \mathrm{J}_{\mathtt{m}} \mathfrak{S}_U \mathfrak{R}_{\check{\chi}} \mathfrak{S}_U^* f_2, f_1\big\rangle_{\mathrm{L}^2}. \tag{5.15}$$

Proof. Taking into account (5.8), by relation (5.9)—with $\hat{K} = I$—we obtain:

$$\begin{aligned}
&\int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h')\, f_2(h') \\
&\qquad = \mathtt{m}(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}} \int \mathrm{d}\mu_G(h')\, \varkappa_U(\chi; h^{-1}g, h')\, f_2(h') \\
&\qquad = \mathtt{m}(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}} \big(\mathfrak{S}_U \mathfrak{R}_{\check{\chi}} \mathfrak{S}_U^* f_2\big)(h^{-1}g).
\end{aligned} \tag{5.16}$$

At this point, relation (5.15) is a straightforward consequence of lemma 5.1. □

Lemma 5.3. *Let* χ *be a vector belonging to* $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$*. Then, for every* $\phi_1 \in \mathcal{H}$*, and for any* ψ_1, ψ_2, ϕ_2 *contained in* $\mathrm{Dom}\big(\hat{D}_U^{-1}\big)$*—setting, as usual,* $\widehat{\phi_j\psi_j} \equiv |\phi_j\rangle\langle\psi_j|$*,* $j = 1, 2$*—we have*

$$\begin{aligned}
(\mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}}(\widehat{\phi_1\psi_1}\widehat{\phi_2\psi_2}))(g) = &\int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h') \\
&\times (\mathfrak{S}_U \widehat{\phi_1\psi_1})(h)\, (\mathfrak{S}_U \widehat{\phi_2\psi_2})(h'), \qquad \forall_{\mu_G} g \in G.
\end{aligned} \tag{5.17}$$

Proof. First observe that

$$\begin{aligned}
&\int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h')\, (\mathfrak{S}_U \widehat{\phi_2\psi_2})(h') \\
&\qquad = \int \mathrm{d}\mu_G(h') \big\langle \hat{D}_U^{-1}\, U(h)^* U(g)\, \hat{D}_U^{-1} \chi, U(h')\, \hat{D}_U^{-1} \chi\big\rangle \big\langle U(h') \hat{D}_U^{-1} \psi_2, \phi_2\big\rangle \\
&\qquad = \langle \psi_2, \chi\rangle \big\langle U(g) \hat{D}_U^{-1} \chi, U(h) \hat{D}_U^{-1} \phi_2\big\rangle,
\end{aligned} \tag{5.18}$$

$\forall\, h, g \in G$, where we have used the fact that ϕ_2 is contained in $\mathrm{Dom}\big(\hat{D}_U^{-1}\big)$. Then, exploiting relation (5.18) and the fact that

$$\int \mathrm{d}\mu_G(h) \big\langle U(g) \hat{D}_U^{-1} \chi, U(h) \hat{D}_U^{-1} \phi_2\big\rangle \big\langle U(h) \hat{D}_U^{-1} \psi_1, \phi_1\big\rangle = \langle \psi_1, \phi_2\rangle \big\langle U(g) \hat{D}_U^{-1} \chi, \phi_1\big\rangle \tag{5.19}$$

—note that $\big\langle U(h) \hat{D}_U^{-1} \psi_1, \phi_1\big\rangle = (\mathfrak{S}_U \widehat{\phi_1\psi_1})(h)$—we find

$$\begin{aligned}
&\int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h')\, (\mathfrak{S}_U \widehat{\phi_1\psi_1})(h)\, (\mathfrak{S}_U \widehat{\phi_2\psi_2})(h') \\
&\qquad = \langle \psi_2, \chi\rangle\, \langle \psi_1, \phi_2\rangle \big\langle U(g) \hat{D}_U^{-1} \chi, \phi_1\big\rangle = \mathfrak{S}_U(\widehat{\phi_1\psi_1}\widehat{\phi_2\psi_2}\widehat{\chi})(g).
\end{aligned} \tag{5.20}$$

The proof is complete. □

As anticipated, the following result can be regarded as a generalization of Lemma 5.3. It will allow us to prove the main result of the paper in a straightforward and transparent way.

Proposition 5.2. *Let* χ *be a vector contained in* $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$*. Then, for any* $f_1, f_2 \in \mathrm{L}^2(G)$*, the following formula holds:*

$$\mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}}\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big) = \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; \cdot, h, h')\, f_1(h)\, f_2(h'). \tag{5.21}$$

Proof. By Lemma 5.3, relation (5.21) holds for any pair of functions f_1, f_2 belonging to the linear span $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H}; U))$ (see (3.4)), which is dense in $\mathcal{R}_U$. Moreover—since $\mathrm{Ker}(\mathfrak{S}_U^*) = \mathcal{R}_U^{\perp}$, and $\mathcal{R}_U$ is an invariant subspace for the complex conjugation $\mathsf{J}_{\mathtt{m}}$ and for the representation $R_{\mathtt{m}}$—for any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$, of which at least one is contained in $\mathcal{R}_U^{\perp}$, we have

$$\big\langle R_{\mathtt{m}}(g) \mathsf{J}_{\mathtt{m}} \mathfrak{S}_U \mathfrak{R}_{\check{\chi}} \mathfrak{S}_U^* f_2, f_1 \big\rangle_{\mathrm{L}^2} = 0. \tag{5.22}$$

Thus, if f_1 and/or f_2 is contained in $\mathcal{R}_U^{\perp}$, recalling relation (5.15), we conclude that

$$\int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; \cdot, h, h')\, f_1(h) f_2(h') = 0. \tag{5.23}$$

Therefore, relation (5.21) is satisfied by f_1, f_2 in the dense linear span $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H}; U)) + \mathcal{R}_U^{\perp}$. In the case where the Hilbert space $\mathcal{H}$ is finite-dimensional (hence, G is unimodular), this linear span actually coincides with $\mathrm{L}^2(G)$ itself and the proof is complete.

Let us assume, instead, that $\dim(\mathcal{H}) = \infty$, and let us prove relation (5.21) for a generic pair of functions in $\mathrm{L}^2(G)$. To this aim, consider first a pair of functions f_1, f_2 of this kind: f_1 is an arbitrary function contained in the dense linear span $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H}; U)) + \mathcal{R}_U^{\perp}$, and f_2 any function belonging to $\mathrm{L}^2(G)$. Next, take a sequence of functions $\{f_{2;n}\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(G)$, contained in $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H}; U)) + \mathcal{R}_U^{\perp}$ and converging (with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$) to f_2. Then, we have

$$\|\cdot\|_{\mathrm{L}^2} \lim_{n\to\infty} \mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}}\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_{2;n}\big)\big) = \mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}}\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big). \tag{5.24}$$

On the other hand, by the first part of the proof and by lemma 5.2, we have that

$$\begin{aligned}
&\lim_{n\to\infty} \big(\mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}}\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_{2;n}\big)\big)\big)(g) \\
&\quad = \lim_{n\to\infty} \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h')\, f_1(h) f_{2;n}(h') \\
&\quad = \lim_{n\to\infty} \big\langle R_{\mathtt{m}}(g) \mathsf{J}_{\mathtt{m}} \mathfrak{S}_U \mathfrak{R}_{\check{\chi}} \mathfrak{S}_U^* f_{2;n}, f_1 \big\rangle_{\mathrm{L}^2} \\
&\quad = \big\langle R_{\mathtt{m}}(g) \mathsf{J}_{\mathtt{m}} \mathfrak{S}_U \mathfrak{R}_{\check{\chi}} \mathfrak{S}_U^* f_2, f_1 \big\rangle_{\mathrm{L}^2} \\
&\quad = \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\, \kappa_U(\chi; g, h, h')\, f_1(h) f_2(h').
\end{aligned} \tag{5.25}$$

From relations (5.24) and (5.25), it descends that formula (5.21) holds true for any pair of functions f_1 contained in the linear span $\big(\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H}; U)) + \mathcal{R}_U^{\perp}\big)$ and $f_2 \in \mathrm{L}^2(G)$. At this point, using this result and a density argument analogous to the one adopted for obtaining it, one proves relation (5.21) for a generic pair of functions in $\mathrm{L}^2(G)$. □

5.3. Formulas for star products

We are now ready to prove the theorem that can be regarded as the main result of the paper. It provides a simple expression for the star product associated with the square integrable projective representation U.

Theorem 5.1. *Let $\{\chi_n\}_{n\in\mathcal{N}}$ be an orthonormal basis in $\mathcal{H}$, contained in the dense linear span* $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$. *Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$f_1 \overset{U}{\star} f_2 = {}_{\|\cdot\|_{\mathrm{L}^2}}\sum_n \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\kappa_U(\chi_n; \cdot, h, h') f_1(h) f_2(h'), \tag{5.26}$$

where the integral kernel $\kappa_U(\chi_n; \cdot, \cdot, \cdot) : G \times G \times G \to \mathbb{C}$ is defined by (5.6), i.e.

$$\kappa_U(\chi_n; g, h, h') := \big\langle U(g)\, \hat{D}_U^{-1}\chi_n, U(h)\, \hat{D}_U^{-1}\, U(h')\, \hat{D}_U^{-1}\chi_n\big\rangle. \tag{5.27}$$

Proof. In order to prove formula (5.26), we can exploit relation (5.21) and the fact that

$$_{\|\cdot\|_{\mathcal{B}_2}}\sum_n \mathfrak{R}_{\widehat{\chi}_n}\hat{A} = \hat{A}, \qquad \forall\, \hat{A} \in \mathcal{B}_2(\mathcal{H}), \tag{5.28}$$

where $\widehat{\chi}_n \equiv |\chi_n\rangle\langle\chi_n|$, as usual; see remark 5.1. Indeed, for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have

$$\begin{aligned}
&_{\|\cdot\|_{\mathrm{L}^2}}\sum_n \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\kappa_U(\chi_n; \cdot, h, h') f_1(h) f_2(h') \\
&\quad = {}_{\|\cdot\|_{\mathrm{L}^2}}\sum_n \mathfrak{S}_U\mathfrak{R}_{\widehat{\chi}_n}\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big) \\
&\quad = \mathfrak{S}_U\, {}_{\|\cdot\|_{\mathcal{B}_2}}\sum_n \mathfrak{R}_{\widehat{\chi}_n}\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big) \\
&\quad = \mathfrak{S}_U\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big).
\end{aligned} \tag{5.29}$$

By definition, the last member of (5.29) is equal to $f_1 \overset{U}{\star} f_2$. □

Remark 5.2. One can readily derive from formula (5.26) various alternative expressions for the star product; in particular, by relation (5.8) we have

$$\begin{aligned}
f_1 \overset{U}{\star} f_2 = {}_{\|\cdot\|_{\mathrm{L}^2}}\sum_n &\int \mathrm{d}\mu_G(h) f_1(h)\mathtt{m}(h, h^{-1}(\cdot))^*\Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \\
&\times \int \mathrm{d}\mu_G(h')\varkappa_U(\chi_n; h^{-1}(\cdot), h') f_2(h').
\end{aligned} \tag{5.30}$$

By the change of variables $h \mapsto gh$ and $h \mapsto h^{-1}$further expressions can be obtained.

Theorem 5.1 has various implications. First of all, it is remarkable that, in the case where G is unimodular, the star product associated with the representation U admits a simple alternative expression.

Corollary 5.1. *Suppose that the l.c.s.c. group G is unimodular. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have*

$$\begin{aligned}
\big(f_1 \overset{U}{\star} f_2\big)(g) &= d_U^{-1} \int \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}(h, h^{-1}g)^*\big(\mathrm{P}_{\mathcal{R}_U} f_2\big)(h^{-1}g) \\
&= d_U^{-1} \int \mathrm{d}\mu_G(h)\big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h)\mathtt{m}(h, h^{-1}g)^* f_2(h^{-1}g) \\
&= d_U^{-1} \int \mathrm{d}\mu_G(h)\big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h)\mathtt{m}(h, h^{-1}g)^*\big(\mathrm{P}_{\mathcal{R}_U} f_2\big)(h^{-1}g), \qquad \forall_{\mu_G} g \in G.
\end{aligned} \tag{5.31}$$

Therefore, for any $f_1, f_2 \in \mathcal{R}_U$, the following formula holds:

$$\big(f_1 \overset{U}{\star} f_2\big)(g) = d_U^{-1} \int \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}(h, h^{-1}g)^* f_2(h^{-1}g), \qquad \forall_{\mu_G} g \in G. \tag{5.32}$$

Proof. Let f_1, f_2 be functions in $\mathrm{L}^2(G)$. Then—using formula (5.26), relation (5.15) and the fact that, being G unimodular, $\mathfrak{R}_{\check{\chi}} = d_U^{-1}\, \mathfrak{R}_{\widehat{\chi}}$—we have

$$\begin{aligned} f_1 \overset{U}{\star} f_2 &= {\scriptstyle\|\cdot\|_{\mathrm{L}^2}} \sum_n \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\kappa_U(\chi_n; \cdot, h, h') f_1(h) f_2(h') \\ &= {\scriptstyle\|\cdot\|_{\mathrm{L}^2}} \sum_n \big\langle R_{\mathtt{m}}(\cdot)\mathsf{J}_{\mathtt{m}}\mathfrak{S}_U \mathfrak{R}_{\check{\chi}_n}\mathfrak{S}_U^* f_2, f_1\big\rangle_{\mathrm{L}^2} \\ &= d_U^{-1} {\scriptstyle\|\cdot\|_{\mathrm{L}^2}} \sum_n \big\langle R_{\mathtt{m}}(\cdot)\mathsf{J}_{\mathtt{m}}\mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}_n}\mathfrak{S}_U^* f_2, f_1\big\rangle_{\mathrm{L}^2}. \end{aligned} \tag{5.33}$$

On the other hand—by virtue of the continuity of the scalar product in $\mathrm{L}^2(G)$ and of the boundedness of the operators $R_{\mathtt{m}}(g)$, $\mathsf{J}_{\mathtt{m}}$ and $\mathfrak{S}_U$, and exploiting relations (5.28) and, then, (5.12) (with $\Delta_G \equiv 1$)—we also have that

$$\begin{aligned} \sum_n \big\langle R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}_n}\mathfrak{S}_U^* f_2, f_1\big\rangle_{\mathrm{L}^2} &= \langle R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\mathfrak{S}_U \mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2} \\ &= \big\langle R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\mathrm{P}_{\mathcal{R}_U} f_2, f_1\big\rangle_{\mathrm{L}^2} \\ &= \int \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}(h, h^{-1}g)^* \big(\mathrm{P}_{\mathcal{R}_U} f_2\big)(h^{-1}g). \end{aligned} \tag{5.34}$$

Relations (5.33) and (5.34) imply that the first of equations (5.31) holds true; the other two are obtained using the fact that $\mathrm{P}_{\mathcal{R}_U}$ is a projector satisfying $R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\mathrm{P}_{\mathcal{R}_U} = \mathrm{P}_{\mathcal{R}_U} R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}$. □

Remark 5.3. We stress that the particularly simple formula (5.32)—differently from formula (5.26)—holds for any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$ of which *at least one* belongs to the (closed) subspace $\mathcal{R}_U$ of $\mathrm{L}^2(G)$, which is the canonical ideal of the H*-algebra $\mathcal{A}_U$, see proposition 4.1. The rhs of (5.32) is a 'twisted convolution' generalizing the standard twisted convolution [4] that appears in the case where G is the group of translations on phase space and U is the projective representation (2.16) (we will examine this case in section 6).

Let us derive another consequence of theorem 5.1. In the case where the group G is compact (hence, unimodular), there is a precise link between the convolution product in $\mathrm{L}^2(G)$ [27] and the star products associated with a realization $\check{G}$ of the unitary dual of G.

Corollary 5.2. *Suppose that the l.c.s.c. group G is compact and that the Haar measure μ_G is normalized as usual for compact groups, i.e. that $\mu_G(G) = 1$. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$\mathrm{L}^2(G) \ni \int \mathrm{d}\mu_G(h)\, f_1(h)\, f_2(h^{-1}(\cdot)) = {\scriptstyle\|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \delta(U)^{-\frac{1}{2}} \big(f_1 \overset{U}{\star} f_2\big). \tag{5.35}$$

Proof. As is well known, since G is compact, the convolution of any pair of functions in $\mathrm{L}^2(G)$ is again a function belonging to $\mathrm{L}^2(G)$. Moreover, from relation (3.9), it follows that ${\scriptstyle\|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \mathrm{P}_{\mathcal{R}_U} f = f, \forall f \in \mathrm{L}^2(G)$; hence—denoting by R the left regular representation of G and by J the complex conjugation

$$\mathrm{L}^2(G) \ni f \mapsto f((\cdot)^{-1})^* \in \mathrm{L}^2(G), \tag{5.36}$$

for any $f_1, f_2 \in \mathrm{L}^2(G)$ we have

$$\begin{aligned}\int \mathrm{d}\mu_G(h)\, f_1(h)\, f_2(h^{-1}g) &= \int \mathrm{d}\mu_G(h) \left({\scriptstyle \|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \mathrm{P}_{\mathcal{R}_U} f_1 \right)(h) f_2(h^{-1}g) \\ &= \left\langle R(g)\mathrm{J} f_2, {\scriptstyle \|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \mathrm{P}_{\mathcal{R}_U} f_1 \right\rangle_{\mathrm{L}^2} \\ &= \sum_{U\in\check{G}} \left\langle R(g)\mathrm{J} f_2, \mathrm{P}_{\mathcal{R}_U} f_1 \right\rangle_{\mathrm{L}^2} \\ &= \sum_{U\in\check{G}} \int \mathrm{d}\mu_G(h) \big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h) f_2(h^{-1}g), \end{aligned} \tag{5.37}$$

for all $g \in G$. On the other hand, by corollary 5.1 we have that

$$\int \mathrm{d}\mu_G(h)\, \big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h) f_2(h^{-1}(\cdot)) = \delta(U)^{-\frac{1}{2}} \big(f_1 \overset{U}{\star} f_2\big), \qquad \forall U \in \check{G}, \tag{5.38}$$

where we recall that $\delta(U)^{-\frac{1}{2}} = d_U$. Moreover, by relations (4.14) and (4.11), for any $f_1, f_2 \in \mathrm{L}^2(G)$ we obtain the following estimate:

$$\begin{aligned}\sum_{U\in\check{G}} \delta(U)^{-1} \big\| f_1 \overset{U}{\star} f_2 \big\|_{\mathrm{L}^2}^2 &= \sum_{U\in\check{G}} \delta(U)^{-1} \big\| \big(\mathrm{P}_{\mathcal{R}_U} f_1\big) \overset{U}{\star} \big(\mathrm{P}_{\mathcal{R}_U} f_2\big) \big\|_{\mathrm{L}^2}^2 \\ &\leqslant \sum_{U\in\check{G}} \delta(U)^{-1} \big\|\mathrm{P}_{\mathcal{R}_U} f_1\big\|_{\mathrm{L}^2}^2 \big\|\mathrm{P}_{\mathcal{R}_U} f_2\big\|_{\mathrm{L}^2}^2 \\ &\leqslant \sum_{U\in\check{G}} \big\|\mathrm{P}_{\mathcal{R}_U} f_1\big\|_{\mathrm{L}^2}^2 \big\|\mathrm{P}_{\mathcal{R}_U} f_2\big\|_{\mathrm{L}^2}^2 \leqslant \|f_1\|_{\mathrm{L}^2}^2 \|f_2\|_{\mathrm{L}^2}^2. \end{aligned} \tag{5.39}$$

Hence, taking into account (4.13), we see that ${\scriptstyle \|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \delta(U)^{-\frac{1}{2}} \big(f_1 \overset{U}{\star} f_2\big)$ is a well-defined element of $\mathrm{L}^2(G)$ and, by (5.38),

$${\scriptstyle \|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \int \mathrm{d}\mu_G(h) \big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h) f_2(h^{-1}(\cdot)) = {\scriptstyle \|\cdot\|_{\mathrm{L}^2}} \sum_{U\in\check{G}} \delta(U)^{-\frac{1}{2}} \big(f_1 \overset{U}{\star} f_2\big). \tag{5.40}$$

At this point, relations (5.37) and (5.40) imply that formula (5.35) holds true. □

We will now prove that it is possible to achieve a simple expression of the $\hat{K}$-deformed star product associated with the representation U, for every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$. Although this result is more general than theorem 5.1—which corresponds to the case where $\hat{K} = I$—we will derive it as a consequence of formula (5.26) for the star product. To this aim, it is useful to observe that, by the definition of the $\hat{K}$-deformed star product and the fact that $\mathfrak{S}_U^* \mathfrak{S}_U = I$, we have

$$\begin{aligned} f_1 \underset{\hat{K}}{\overset{U}{\star}} f_2 &:= \mathfrak{S}_U\big(\mathfrak{S}_U^* f_1\, \hat{K}\, \mathfrak{S}_U^* f_2\big) \\ &= \mathfrak{S}_U\big(\mathfrak{S}_U^* f_1\, \mathfrak{S}_U^*\big(\mathfrak{S}_U\big(\hat{K}\, \mathfrak{S}_U^* f_2\big)\big)\big) = f_1 \overset{U}{\star} \big(\mathfrak{S}_U\big(\hat{K}\, \mathfrak{S}_U^* f_2\big)\big). \end{aligned} \tag{5.41}$$

Moreover, for every bounded operator $\hat{K}$ in $\mathcal{H}$ and for every vector χ contained in $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$, let us define an integral kernel $\kappa_U(\hat{K}, \chi; \cdot, \cdot, \cdot) : G \times G \times G \to \mathbb{C}$ by setting

$$\begin{aligned} \kappa_U(\hat{K}, \chi; g, h, h') &:= \big\langle \hat{D}_U^{-1}\, U(h)^* U(g)\, \hat{D}_U^{-1} \chi, \hat{K}\, U(h')\, \hat{D}_U^{-1} \chi \big\rangle \\ &= \mathtt{m}(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}}\, \varkappa_U(\hat{K}, \chi; h^{-1}g, h'). \end{aligned} \tag{5.42}$$

Comparing this definition with (5.6), it is clear that $\kappa_U(\chi; g, h, h') \equiv \kappa_U(I, \chi; g, h, h')$.

Corollary 5.3. *Let $\hat{K}$ be a bounded operator in $\mathcal{H}$ and $\{\chi_n\}_{n\in\mathcal{N}}$ an orthonormal basis contained in the dense linear span* $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$*. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 =_{\|\cdot\|_{\mathrm{L}^2}} \sum_n \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\kappa_U(\hat{K}, \chi_n; \cdot, h, h') f_1(h) f_2(h'). \tag{5.43}$$

Proof. Taking into account relation (5.41), we can apply formula (5.26) for the (standard) star product, and next we use relation (5.15), thus getting

$$\begin{aligned} f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 &=_{\|\cdot\|_{\mathrm{L}^2}} \sum_n \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\kappa_U(\chi_n; \cdot, h, h') f_1(h)\big(\mathfrak{S}_U\big(\hat{K}\mathfrak{S}_U^* f_2\big)\big)(h') \\ &=_{\|\cdot\|_{\mathrm{L}^2}} \sum_n \big\langle R_{\mathtt{m}}(\cdot)\mathtt{J}_{\mathtt{m}}\mathfrak{S}_U\mathfrak{R}_{\check{\chi}_n}\mathfrak{S}_U^*\big(\mathfrak{S}_U\big(\hat{K}\mathfrak{S}_U^* f_2\big)\big), f_1\big\rangle_{\mathrm{L}^2} \\ &=_{\|\cdot\|_{\mathrm{L}^2}} \sum_n \big\langle R_{\mathtt{m}}(\cdot)\mathtt{J}_{\mathtt{m}}\big(\mathfrak{S}_U\mathfrak{R}_{\check{\chi}_n}\big(\hat{K}\mathfrak{S}_U^* f_2\big)\big), f_1\big\rangle_{\mathrm{L}^2}. \end{aligned} \tag{5.44}$$

From (5.44), by virtue of relations (5.14), (5.9) and (5.42), it follows that

$$\begin{aligned} f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 &=_{\|\cdot\|_{\mathrm{L}^2}} \sum_n \int \mathrm{d}\mu_G(h) f_1(h)\mathtt{m}(h, h^{-1}g)^*\Delta_G(h^{-1}(\cdot))^{\frac{1}{2}}\big(\mathfrak{S}_U\mathfrak{R}_{\check{\chi}_n}\mathfrak{L}_{\hat{K}}\mathfrak{S}_U^* f_2\big)(h^{-1}(\cdot)) \\ &=_{\|\cdot\|_{\mathrm{L}^2}} \sum_n \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\kappa_U(\hat{K}, \chi_n; \cdot, h, h') f_1(h) f_2(h'). \end{aligned} \tag{5.45}$$

The proof is complete. □

Formula (5.43) assumes a remarkably simple form in the special case where the carrier Hilbert space $\mathcal{H}$ of the representation U is finite-dimensional (so that the l.c.s.c. group G must be unimodular; see the last assertion of remark 2.1). Indeed, one easily derives the following result.

Corollary 5.4. *Suppose that the Hilbert space $\mathcal{H}$, where the square integrable representation U acts, is finite-dimensional. Then, for any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 = d_U^{-3} \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\ \mathrm{tr}(U(\cdot)^* U(h)\, \hat{K}\, U(h'))\, f_1(h) f_2(h'). \tag{5.46}$$

Remark 5.4. Assume that G is a compact—in particular, a finite—group and U is a (irreducible) unitary representation. In this case, formula (5.46) reads:

$$f_1 \overset{U}{\star} f_2 = \delta(U)^{\frac{3}{2}} \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\ \mathtt{C}_U((\cdot)^{-1}hh')\, f_1(h) f_2(h'), \tag{5.47}$$

where $\mathtt{C}_U : G \to \mathbb{C}$ is the character of the finite-dimensional representation U, i.e. $\mathtt{C}_U(g) := \mathrm{tr}(U(g))$. Then, since $\mathfrak{S}_U I = \delta(U)^{\frac{1}{2}}\, \mathtt{C}_U((\cdot)^{-1})$, the obvious equation $(\mathfrak{S}_U I) \overset{U}{\star} (\mathfrak{S}_U I) = \mathfrak{S}_U I$ translates into the following relation for the character $\mathtt{C}_U$:

$$\mathtt{C}_U(g) = \delta(U)^2 \int \mathrm{d}\mu_G(h) \int \mathrm{d}\mu_G(h')\ \mathtt{C}_U(ghh')\, \mathtt{C}_U(h^{-1})\, \mathtt{C}_U((h')^{-1}). \tag{5.48}$$

Thus, we recover results previously found in [22].

6. Applications

In this section, we will consider two simple—but extremely significant—applications of the theory developed in sections 3–5. We will first consider the case of a square integrable—genuinely projective—representation of a unimodular group, i.e. the group of translations on phase space. The analysis of this case leads to the Groenewold–Moyal star product, i.e. the prototype of the star product. Next, we will study a case where square integrable unitary representations of a group which is *not* unimodular—the one-dimensional affine group—are involved. As already mentioned, this group is at the base of wavelet analysis.

6.1. The group of translations on phase space

Let us consider the group of translations on the $(1+1)$-dimensional phase space, namely, the additive group $\mathbb{R}\times\mathbb{R}$ (the extension to the $(n+n)$-dimensional case is straightforward). As is well known (see, e.g., [39]), the map $\mathbb{R}\times\mathbb{R}\ni(q,p)\mapsto U(q,p)\in\mathcal{U}(\mathrm{L}^2(\mathbb{R}))$, defined by

$$U(q,p):=\exp(\mathrm{i}(p\hat{q}-q\hat{p}))=\mathrm{e}^{-\frac{\mathrm{i}}{2}qp}\exp(\mathrm{i}p\hat{q})\exp(-\mathrm{i}q\hat{p})=\mathrm{e}^{\frac{\mathrm{i}}{2}qp}\exp(-\mathrm{i}q\hat{p})\exp(\mathrm{i}p\hat{q}),\tag{6.1}$$

$q,p\in\mathbb{R}$—where $\hat{q}$, $\hat{p}$ are the standard position and momentum operators—is a projective representation of the unimodular group $\mathbb{R}\times\mathbb{R}$, representation which we will call (with a slight abuse of terminology) *Weyl system*. The Weyl system—as already observed in section 2—is a square integrable representation. It 'encodes' the canonical commutation relations of quantum mechanics (in the integrated form), as shown by the last two members of (6.1).

The (generalized) Wigner transform generated by the Weyl system is not the standard Wigner transform but the so-called *Fourier–Wigner transform* [40]. In fact, it turns out that these maps are related by the *symplectic Fourier transform*, i.e. by the unitary operator $\mathcal{F}_{\mathrm{sp}}:\mathrm{L}^2(\mathbb{R}\times\mathbb{R})\to\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ determined by

$$(\mathcal{F}_{\mathrm{sp}}f)(q,p)=\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}}f(q',p')\,\mathrm{e}^{\mathrm{i}(qp'-pq')}\,\mathrm{d}q'\,\mathrm{d}p',\qquad\forall f\in\mathrm{L}^1(\mathbb{R}\times\mathbb{R})\cap\mathrm{L}^2(\mathbb{R}\times\mathbb{R}).\tag{6.2}$$

Recall that $\mathcal{F}_{\mathrm{sp}}$ enjoys the remarkable property of being both unitary and self-adjoint: $\mathcal{F}_{\mathrm{sp}}=\mathcal{F}_{\mathrm{sp}}^*$, $\mathcal{F}_{\mathrm{sp}}^2=I$.

As already mentioned in section 2, $(2\pi)^{-1}\mathrm{d}q\,\mathrm{d}p$ is the Haar measure on $\mathbb{R}\times\mathbb{R}$ normalized in agreement with the Weyl system U. Then, in this case, the generalized Wigner transform $\mathfrak{S}_U$ is the isometry from $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}))$ into $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})\equiv\mathrm{L}^2(\mathbb{R}\times\mathbb{R},(2\pi)^{-1}\mathrm{d}q\,\mathrm{d}p;\mathbb{C})$ determined by

$$(\mathfrak{S}_U\hat{\rho})(q,p)=\mathrm{tr}(U(q,p)^*\hat{\rho}),\qquad\forall\hat{\rho}\in\mathcal{B}_1(\mathrm{L}^2(\mathbb{R})).\tag{6.3}$$

The multiplier $\mathtt{m}:(\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R})\to\mathbb{T}$ associated with U is of the form

$$\mathtt{m}(q,p;q',p')=\exp\left(\frac{\mathrm{i}}{2}(qp'-pq')\right).\tag{6.4}$$

Therefore, according to formulas (3.11) and (3.13), the generalized Wigner transform $\mathfrak{S}_U$ intertwines the unitary representation $U\vee U:\mathbb{R}\times\mathbb{R}\to\mathcal{U}(\mathcal{B}_2(\mathrm{L}^2(\mathbb{R})))$ with the representation $\mathcal{T}_{\mathtt{m}}:\mathbb{R}\times\mathbb{R}\to\mathcal{U}(\mathrm{L}^2(\mathbb{R}\times\mathbb{R}))$ defined by

$$(\mathcal{T}_{\mathtt{m}}(q,p)f)(q',p')=\mathrm{e}^{-\mathrm{i}(qp'-pq')}f(q',p'),\qquad\forall f\in\mathrm{L}^2(\mathbb{R}\times\mathbb{R}).\tag{6.5}$$

Moreover, $\mathfrak{S}_U$ intertwines the involution $\mathfrak{J}$ in $\mathcal{B}_2(\mathcal{H})$ with the complex conjugation $\mathsf{J}\equiv\mathsf{J}_{\mathtt{m}}$ that, in this case—as the reader may readily check—takes the following form:

$$(\mathsf{J}f)(q,p)=f(-q,-p)^*,\qquad\forall f\in\mathrm{L}^2(\mathbb{R}\times\mathbb{R}).\tag{6.6}$$

As anticipated, the *standard Wigner transform*—we will denote it by $\mathfrak{T}$—is the isometry obtained composing the isometry $\mathfrak{S}_U$, determined by (6.3), with the symplectic Fourier transform (see [23]):

$$\mathfrak{T} := \mathcal{F}_{\mathrm{sp}}\, \mathfrak{S}_U : \mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \to \mathrm{L}^2(\mathbb{R}\times\mathbb{R}). \tag{6.7}$$

It is clear that the isometry $\mathfrak{T}$ intertwines the representation $U \vee U$ with the unitary representation $\mathcal{V} : \mathbb{R}\times\mathbb{R} \to \mathcal{U}(\mathrm{L}^2(\mathbb{R}\times\mathbb{R}))$ defined by $\mathcal{V}(q,p) := \mathcal{F}_{\mathrm{sp}}\mathcal{T}_{\mathtt{m}}(q,p)\mathcal{F}_{\mathrm{sp}}$, $\forall\,(q,p)\in\mathbb{R}\times\mathbb{R}$; as the reader may easily check, explicitly, we have

$$(\mathcal{V}(q,p)f)(q',p') = f(q'-q,p'-p), \qquad \forall\, f\in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}). \tag{6.8}$$

Thus, the representation $\mathcal{V}$ acts by simply translating functions on phase space. It is also a remarkable fact—see [41]—that $\mathrm{Ran}(\mathfrak{T}) = \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$; equivalently, $\mathcal{R}_U \equiv \mathrm{Ran}(\mathfrak{S}_U) = \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ (this fact can be verified deducing the integral kernel of the Hilbert–Schmidt operator $\mathfrak{S}_U^* f$, for a generic $f\in\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$, and observing that $\mathrm{Ker}(\mathfrak{S}_U^*) = \{0\}$). Therefore, the standard Wigner transform $\mathfrak{T}$ and its adjoint $\mathfrak{T}^*$, the *standard Weyl map,* are both unitary operators.

Let us now study the star product in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ induced by the Weyl system U. Recalling theorem 5.1, and taking into account the fact that, in this case, $\mathcal{R}_U = \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ (and $d_U = 1$), we have

$$\begin{aligned}\big(f_1 \overset{U}{\star} f_2\big)(q,p) &= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} f_1(q',p')\mathtt{m}(q,p;q-q',p-p')^*\, f_2(q-q',p-p')\,\mathrm{d}q'\,\mathrm{d}p'\\ &= \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} f_1(q',p')\, f_2(q-q',p-p')\exp\Big(\frac{\mathrm{i}}{2}(qp'-pq')\Big)\,\mathrm{d}q'\,\mathrm{d}p',\end{aligned} \tag{6.9}$$

$\forall\, f_1, f_2 \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$. Thus, the star product associated with the Weyl system is nothing but the *twisted convolution* of functions [4] (see also [40, 42]). According to the results of section 4, $(\mathrm{L}^2(\mathbb{R}\times\mathbb{R}), \overset{U}{\star}, \mathsf{J})$ is a proper H*-algebra and $\mathfrak{S}_U : \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ is an isomorphism of H*-algebras.

The unitary operators $\mathfrak{T}, \mathfrak{T}^*$ induce another star product of functions

$$(\cdot)\circledast(\cdot) : \mathrm{L}^2(\mathbb{R}\times\mathbb{R})\times\mathrm{L}^2(\mathbb{R}\times\mathbb{R}) \ni (f_1,f_2) \mapsto \mathfrak{T}((\mathfrak{T}^* f_1)(\mathfrak{T}^* f_2)) \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}), \tag{6.10}$$

namely the *twisted product* (see [4]). Using the fact that $\mathfrak{T} = \mathcal{F}_{\mathrm{sp}}\,\mathfrak{S}_U$ and $\mathfrak{T}^* = \mathfrak{S}_U^*\,\mathcal{F}_{\mathrm{sp}}$, we obtain that

$$f_1 \circledast f_2 = \mathcal{F}_{\mathrm{sp}}((\mathcal{F}_{\mathrm{sp}} f_1) \overset{U}{\star} (\mathcal{F}_{\mathrm{sp}} f_2)). \tag{6.11}$$

From this relation, by an explicit calculation, one finds that, for any $f_1, f_2 \in \mathrm{L}^1(\mathbb{R}\times\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$,

$$\begin{aligned}&(f_1\circledast f_2)(q,p)\\ &\quad= \frac{1}{\pi^2}\int_{\mathbb{R}\times\mathbb{R}}\mathrm{d}q'\,\mathrm{d}p'\int_{\mathbb{R}\times\mathbb{R}}\mathrm{d}q''\,\mathrm{d}p''\,\theta(q,p;q',p';q'',p'')f_1(q',p')f_2(q'',p''),\end{aligned} \tag{6.12}$$

where we have set

$$\theta(q,p;q',p';q'',p'') := \exp(\mathrm{i}2(qp'-pq'+q'p''-p'q''+q''p-p''q)). \tag{6.13}$$

The function $\theta : (\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R}) \to \mathbb{T}$ is the celebrated *Groenewold–Moyal kernel*. The symplectic Fourier transform intertwines the complex conjugation J with the standard complex conjugation in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$: $\mathcal{F}_{\mathrm{sp}}\mathsf{J}\mathcal{F}_{\mathrm{sp}} f = f^*$. Therefore, $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ endowed with the twisted product and with the standard complex conjugation is again a proper H*-algebra. This fact seems to have been noted for the first time by Pool [41].

6.2. The one-dimensional affine group

Let us consider, now, the *one-dimensional affine group*, namely, the semi-direct product group $G = \mathbb{R} \rtimes \mathbb{R}_*^+$, where $\mathbb{R}_*^+$ is the subgroup of *dilations*, i.e. $\mathbb{R}_*^+$ is the group of strictly positive real numbers (we will denote by $\mathbb{R}_*^-$ the set of strictly negative real numbers) which acts multiplicatively on $\mathbb{R}$. Thus, G consists of the topological space $\mathbb{R} \times \mathbb{R}_*^+$, endowed with the composition law $(a, r)\,(a', r') = (a + ra', rr')$, $a \in \mathbb{R}$, $r \in \mathbb{R}_*^+$. This group is not unimodular. A pair μ_{L}, μ_{R} of—left and right, respectively—conjugated Haar measures on G ($\int_G f(g)\,\mathrm{d}\mu_{\mathrm{L}}(g) = \int_G f(g^{-1})\,\mathrm{d}\mu_{\mathrm{R}}(g)$) are given by $\mathrm{d}\mu_{\mathrm{L}}(a, r) = r^{-2}\,\mathrm{d}a\,\mathrm{d}r$, $\mathrm{d}\mu_{\mathrm{R}}(a, r) = r^{-1}\,\mathrm{d}a\,\mathrm{d}r$, $a \in \mathbb{R}$, $r \in \mathbb{R}_*^+$. Hence, the modular function Δ_G on G is given by $\Delta_G(a, r) = r^{-1}$, $\forall\, a \in \mathbb{R}$, $\forall\, r \in \mathbb{R}_*^+$. As already recalled in section 2, this group is at the base of the theory of the wavelet transform. For the sake of completeness, we will come back to this point later on. It is also worth mentioning that the quantization–dequantization theory based on the affine group has been studied by Aslaksen and Klauder [43], who obtained the Wigner and Weyl maps associated with the representations of this group. However, they did not consider the concept of star product.

Using Mackey's little group method for classifying the irreducible representations of semi-direct product groups with an Abelian normal factor (see [26]), and the results of [44] on the characterization of square integrable representations of the groups of this type, one finds out that the affine group G admits a maximal set of mutually unitarily inequivalent, square integrable, irreducible unitary representations consisting of two elements: $\{U^{(-)} : G \to \mathcal{U}(\mathrm{L}^2(\mathbb{R}_*^-)), U^{(+)} : G \to \mathcal{U}(\mathrm{L}^2(\mathbb{R}_*^+))\}$. These two unitary representations are defined by

$$\begin{aligned} &(U^{(-)}(a, r)\varphi^{(-)})(x) := r^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}ax}\varphi^{(-)}(rx), \\ &a \in \mathbb{R}, \quad r \in \mathbb{R}_*^+, \quad x \in \mathbb{R}_*^-, \quad \varphi^{(-)} \in \mathrm{L}^2(\mathbb{R}_*^-), \end{aligned} \tag{6.14}$$

$$\begin{aligned} &(U^{(+)}(a, r)\varphi^{(+)})(x) := r^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}ax}\varphi^{(+)}(rx), \\ &a \in \mathbb{R}, \quad r \in \mathbb{R}_*^+, \quad x \in \mathbb{R}_*^+, \quad \varphi^{(+)} \in \mathrm{L}^2(\mathbb{R}_*^+), \end{aligned} \tag{6.15}$$

where the Hilbert space $\mathrm{L}^2(\mathbb{R}_*^\pm)$ is of course defined considering the restriction to $\mathbb{R}_*^\pm$ of the Lebesgue measure on $\mathbb{R}$. Moreover, by the results of [44], the Duflo–Moore operator $\hat{D}_{(\pm)}$ associated with the representation $U^{(\pm)}$—and normalized according to μ_{L}—is the unbounded multiplication operator (defined on its natural domain) by the function $\mathbb{R}_*^\pm \ni x \mapsto \sqrt{2\pi/|x|}$.

The representations $U^{(-)}$, $U^{(+)}$ are unitarily inequivalent, but they are intertwined by the antiunitary operator $\mathrm{L}^2(\mathbb{R}_*^-) \ni \varphi \mapsto \varphi(-(\cdot))^* \in \mathrm{L}^2(\mathbb{R}_*^+)$. We will denote by $\mathfrak{S}_{(-)}$ and $\mathfrak{S}_{(+)}$, respectively, the associated Wigner maps. These maps are isometries that intertwine the unitary representations $U^{(-)} \vee U^{(-)}$ and $U^{(+)} \vee U^{(+)}$, respectively, with the two-sided regular representation $\mathcal{T}$ of $\mathbb{R} \rtimes \mathbb{R}_*^+$, representation which is defined by

$$(\mathcal{T}(a, r)f)(a', r') := r^{-\frac{1}{2}}\, f(r^{-1}(a' - a + r'a), r'), \qquad \forall\, f \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}}). \tag{6.16}$$

The standard involutions $\mathfrak{J}_{(-)}$, $\mathfrak{J}_{(+)}$ in the Hilbert–Schmidt spaces $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}_*^-))$, $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}_*^+))$ are intertwined by the Wigner maps $\mathfrak{S}_{(-)}$ and $\mathfrak{S}_{(+)}$, respectively, with the map

$$\mathsf{J} : \mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}}) \to \mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}}), \tag{6.17}$$

which is the complex conjugation defined by

$$(\mathsf{J}f)(a, r) = r^{\frac{1}{2}}\, f(-r^{-1}a, r^{-1})^*, \qquad \forall\, f \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}}). \tag{6.18}$$

The explicit form of the Weyl map $\mathfrak{S}^*_{(\pm)} : \mathrm{L}^2(G) \to \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}^\pm_*))$ can be easily obtained applying formula (3.22). Indeed, for every function $\mathsf{f} : G \to \mathbb{C}$ in $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ and every vector $\varphi^{(\pm)}$ in $\mathrm{Dom}\big(\hat{D}^{-1}_{(\pm)}\big)$, we have

$$\begin{aligned}\big(\big(\mathfrak{S}^*_{(\pm)}\mathsf{f}\big)\varphi^{(\pm)}\big)(x) &= \int_G \mathsf{f}(a,r)\big(U^{(\pm)}(a,r)\hat{D}^{-1}_{(\pm)}\varphi^{(\pm)}\big)(x)\mathrm{d}\mu_{\mathrm{L}}(a,r)\\ &= \int_G \mathsf{f}(a,r)\sqrt{r}\,\mathrm{e}^{\mathrm{i}ax}\sqrt{\frac{r|x|}{2\pi}}\varphi^{(\pm)}(rx)\,\mathrm{d}\mu_{\mathrm{L}}(a,r), \qquad \text{for a.a. } x \in \mathbb{R}^\pm_*.\end{aligned} \tag{6.19}$$

Next, by virtue of Fubini's theorem and of a change of variables ($r \mapsto x^{-1}y$, with $x, y \in \mathbb{R}^\pm_*$), we get

$$\begin{aligned}\big(\big(\mathfrak{S}^*_{(\pm)}\mathsf{f}\big)\varphi^{(\pm)}\big)(x) &= \int_{\mathbb{R}^\pm_*} \mathrm{d}y\,\frac{\sqrt{|x|}\varphi^{(\pm)}(y)}{|y|}\int_{\mathbb{R}}\frac{\mathrm{d}a}{\sqrt{2\pi}}\,\mathsf{f}(a,x^{-1}y)\,\mathrm{e}^{\mathrm{i}ax}\\ &= \int_{\mathbb{R}^\pm_*} \varsigma^{(\pm)}_{\mathsf{f}}(x,y)\varphi^{(\pm)}(y)\,\mathrm{d}y,\end{aligned} \tag{6.20}$$

for a.a. $x \in \mathbb{R}^\pm_*$, where—for every $f \in \mathrm{L}^2(G)$—the integral kernel $\varsigma^{(\pm)}_f(\cdot,\cdot) : \mathbb{R}^\pm_* \times \mathbb{R}^\pm_* \to \mathbb{C}$ is defined by

$$\varsigma^{(\pm)}_f(x,y) := |x|^{\frac{1}{2}}|y|^{-1}(\mathcal{F}_1 f)(-x, x^{-1}y), \qquad x, y \in \mathbb{R}^\pm_*, \tag{6.21}$$

with $\mathcal{F}_1$ denoting the Fourier transform with respect to the first variable. This result—by the well-known essential uniqueness of the inducing kernel of a Hilbert–Schmidt operator—implies that $\varsigma^{(\pm)}_{\mathsf{f}}(\cdot,\cdot)$ is the integral kernel associated with the Hilbert–Schmidt operator $\mathfrak{S}^*_{(\pm)}\mathsf{f}$ in $\mathrm{L}^2(\mathbb{R}^\pm_*)$, for every $\mathsf{f} \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$; hence, we have that

$$\begin{aligned}\|\mathfrak{S}^*_{(\pm)}\mathsf{f}\|^2_{\mathcal{B}_2} &= \int_{\mathbb{R}^\pm_*}\mathrm{d}x\int_{\mathbb{R}^\pm_*}\mathrm{d}y\,\frac{|x|}{y^2}|(\mathcal{F}_1\mathsf{f})(-x,x^{-1}y)|^2\\ &= \int_{\mathbb{R}^\pm_*}\mathrm{d}x\int_{\mathbb{R}^+_*}\frac{\mathrm{d}r}{r^2}|(\mathcal{F}_1\mathsf{f})(-x,r)|^2\\ &\leqslant \int_{\mathbb{R}}\mathrm{d}x\int_{\mathbb{R}^+_*}\frac{\mathrm{d}r}{r^2}|(\mathcal{F}_1\mathsf{f})(-x,r)|^2 = \|\mathsf{f}\|^2_{\mathrm{L}^2}.\end{aligned} \tag{6.22}$$

Of course, what we have found—i.e. $\|\mathfrak{S}^*_{(\pm)}\mathsf{f}\|^2_{\mathcal{B}_2} \leqslant \|\mathsf{f}\|^2_{\mathrm{L}^2}$—is coherent with the fact that the Weyl map $\mathfrak{S}^*_{(\pm)}$ is a partial isometry. Now, let f be a *generic* function in $\mathrm{L}^2(G)$ and $\{\mathsf{f}_n\}_{n\in\mathbb{N}}$ a sequence in $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ such that $\lim_{n\to\infty}\|f - \mathsf{f}_n\|_{\mathrm{L}^2} = 0$. Then, the sequence $\{\mathfrak{S}^*_{(\pm)}\mathsf{f}_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_2(\mathcal{H})$ converges to $\mathfrak{S}^*_{(\pm)}f$; equivalently, the sequence $\big\{\varsigma^{(\pm)}_{\mathsf{f}_n}\big\}_{n\in\mathbb{N}}$ converges in $\mathrm{L}^2(\mathbb{R}^\pm_* \times \mathbb{R}^\pm_*)$ to the integral kernel of the Hilbert–Schmidt operator $\mathfrak{S}^*_{(\pm)}f$, kernel which for the moment is still 'unknown'. But, arguing as in (6.22), we see that the function $\varsigma^{(\pm)}_f$ belongs to $\mathrm{L}^2(\mathbb{R}^\pm_* \times \mathbb{R}^\pm_*)$ and

$$\big\|\varsigma^{(\pm)}_f - \varsigma^{(\pm)}_{\mathsf{f}_n}\big\|^2_{\mathrm{L}^2(\mathbb{R}^\pm_*\times\mathbb{R}^\pm_*)} = \int_{\mathbb{R}^\pm_*}\mathrm{d}x\int_{\mathbb{R}^\pm_*}\mathrm{d}y\,\frac{|x|}{y^2}|(\mathcal{F}_1(f-\mathsf{f}_n))(-x,x^{-1}y)|^2 \leqslant \|f - \mathsf{f}_n\|^2_{\mathrm{L}^2}. \tag{6.23}$$

It follows that the integral kernel of $\mathfrak{S}^*_{(\pm)}f$ is $\varsigma^{(\pm)}_f$ for *every* $f \in \mathrm{L}^2(G)$. Moreover, we have that

$$\begin{aligned}\|\mathfrak{S}^*_{(-)}f\|^2_{\mathcal{B}_2} + \|\mathfrak{S}^*_{(+)}f\|^2_{\mathcal{B}_2} &= \int_{\mathbb{R}^-_*}\mathrm{d}x\int_{\mathbb{R}^+_*}\frac{\mathrm{d}r}{r^2}|(\mathcal{F}_1 f)(-x,r)|^2 + \int_{\mathbb{R}^+_*}\mathrm{d}x\int_{\mathbb{R}^+_*}\frac{\mathrm{d}r}{r^2}|(\mathcal{F}_1 f)(-x,r)|^2\\ &= \int_G |f(a,r)|^2\,r^{-2}\mathrm{d}a\,\mathrm{d}r = \|f\|^2_{\mathrm{L}^2}, \qquad \forall\, f \in \mathrm{L}^2(G).\end{aligned} \tag{6.24}$$

Therefore, denoting by $\mathcal{R}_{(\pm)}$ the range of the Wigner map $\mathfrak{S}_{(\pm)}$ (we know that $\mathcal{R}_{(-)} \perp \mathcal{R}_{(+)}$, see remark 3.1)—since $\mathcal{R}_{(\pm)} = \mathrm{Ker}\big(\mathfrak{S}^*_{(\pm)}\big)^{\perp}$—the following relation must hold: $\mathrm{L}^2(G) = \mathcal{R}_{(-)} \oplus \mathcal{R}_{(+)}$.

Let us now consider the star products in $\mathrm{L}^2(G)$ associated with the square integrable representations $U^{(-)}$ and $U^{(+)}$. By definition—see (4.1)—we have

$$f_1 \overset{(\pm)}{\star} f_2 := \mathfrak{S}_{(\pm)}\big(\big(\mathfrak{S}^*_{(\pm)} f_1\big)\big(\mathfrak{S}^*_{(\pm)} f_2\big)\big), \qquad \forall\, f_1, f_2 \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}^+_*, \mu_{\mathrm{L}}). \tag{6.25}$$

Exploiting the results of section 5, we can provide explicit formulas for these star products. Let $\{\chi_n^{(\pm)}\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathrm{L}^2(\mathbb{R}^\pm_*)$ contained in $\mathrm{Dom}(\hat{D}^{-2}_{(\pm)})$, i.e., such that $(\mathbb{R}^\pm_* \ni x \mapsto |x|\ \chi_n^{(\pm)}(x)) \in \mathrm{L}^2(\mathbb{R}^\pm_*)$. For instance, one can choose the Laguerre functions $\chi_n^{(\pm)} : \mathbb{R}^\pm_* \ni x \mapsto L_{n-1}(|x|)\,\mathrm{e}^{-|x|/2}$, where L_k, $k = 0, 1, 2, \ldots$, is the Laguerre polynomial of order k. According to the main result of section 5—see theorem 5.1—we have

$$f_1 \overset{(\pm)}{\star} f_2 = \|\cdot\|_{\mathrm{L}^2} \sum_{n\in\mathbb{N}} \int_G \mathrm{d}\mu_{\mathrm{L}}(a,r) \int_G \mathrm{d}\mu_{\mathrm{L}}(a',r')\kappa_{(\pm)}(\chi_n^{(\pm)}; \cdot,\cdot; a, r; a', r') \times f_1(a,r) f_2(a',r'), \tag{6.26}$$

where the integral kernel $\kappa_{(\pm)}(\chi_n^{(\pm)}; \cdot,\cdot; \cdot,\cdot; \cdot,\cdot) : G\times G\times G \to \mathbb{C}$ is defined by

$$\kappa_{(\pm)}\big(\chi_n^{(\pm)}; a_1, r_1; a_2, r_2; a_3, r_3\big) := \big\langle U^{(\pm)}(a_1,r_1)\hat{D}^{-1}_{(\pm)}\chi_n^{(\pm)}, U^{(\pm)}(a_2,r_2)\hat{D}^{-1}_{(\pm)}U^{(\pm)}(a_3,r_3)\hat{D}^{-1}_{(\pm)}\chi_n^{(\pm)}\big\rangle. \tag{6.27}$$

Recalling the explicit form of the the Duflo–Moore operators $\hat{D}_{(\pm)}$, we have:

$$\begin{aligned}\kappa_{(\pm)}\big(\chi_n^{(\pm)}; a_1, r_1; a_2, r_2; a_3, r_3\big) &= \frac{r_2\sqrt{r_3}}{r_1}\big\langle \hat{D}^{-1}_{(\pm)}\chi_n^{(\pm)}, \hat{D}^{-2}_{(\pm)}U^{(\pm)}(-(a_1-a_2-r_2a_3)/r_1, r_2r_3/r_1)\chi_n^{(\pm)}\big\rangle \\ &= \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} \frac{r_3}{2\pi}\Lambda_n^{(\pm)}((a_1-a_2-r_2a_3)/r_1, r_2r_3/r_1),\end{aligned} \tag{6.28}$$

where the function $\Lambda_n^{(\pm)} : \mathbb{R}\times\mathbb{R}^+_* \to \mathbb{C}$ is defined by

$$\Lambda_n^{(\pm)}(\alpha,\varrho) := \mathcal{F}\big(|\cdot|^{\frac{3}{2}}\ \breve{\chi}_n^{(\pm)}(\cdot)^*\breve{\chi}_n^{(\pm)}(\varrho(\cdot))\big)(\alpha), \qquad \alpha\in\mathbb{R}, \quad \varrho\in\mathbb{R}^+_*, \tag{6.29}$$

with $\mathcal{F} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ denoting the Fourier transform and $\breve{\chi}_n^{(\pm)} \in \mathrm{L}^2(\mathbb{R})$ the function

$$\breve{\chi}_n^{(\pm)}(x) = \chi_n^{(\pm)}(x), \qquad \text{for } x\in\mathbb{R}^\pm_*, \qquad \breve{\chi}_n^{(\pm)}(x) = 0, \qquad \text{otherwise}, \tag{6.30}$$

i.e. $\breve{\chi}_n^{(\pm)}$ is the image of $\chi_n^{(\pm)}$ via the natural immersion of $\mathrm{L}^2(\mathbb{R}^\pm_*)$ into $\mathrm{L}^2(\mathbb{R})$. In conclusion, the triples

$$\mathcal{A}_{(-)} \equiv \big(\mathrm{L}^2\big(\mathbb{R}\times\mathbb{R}^+_*, \mu_{\mathrm{L}}\big), (\cdot)\overset{(-)}{\star}(\cdot), \mathsf{J}\big) \qquad \text{and} \qquad \mathcal{A}_{(+)} \equiv \big(\mathrm{L}^2\big(\mathbb{R}\times\mathbb{R}^+_*, \mu_{\mathrm{L}}\big), (\cdot)\overset{(+)}{\star}(\cdot), \mathsf{J}\big) \tag{6.31}$$

are H*-algebras. The mutually orthogonal subspaces $\mathcal{R}_{(-)}$ and $\mathcal{R}_{(+)}$ of $\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^+_*, \mu_{\mathrm{L}})$ are, respectively, the canonical and the annihilator ideals in the standard decomposition of the H*-algebra $\mathcal{A}_{(-)}$, while they are, respectively, the annihilator and the canonical ideals for $\mathcal{A}_{(+)}$. It is clear that one may endow $\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^+_*, \mu_{\mathrm{L}})$ with the structure of a *proper* H*-algebra by considering the star product

$$f_1 \star f_2 := \big(f_1 \overset{(-)}{\star} f_2\big) + \big(f_1 \overset{(+)}{\star} f_2\big). \tag{6.32}$$

Let us now clarify the link with the standard wavelet transform. To this aim, let us consider the unitary representation $\widetilde{U} : G \to \mathcal{U}(\mathrm{L}^2(\mathbb{R}))$ defined as follows. Taking into account the orthogonal sum decomposition $L^2(\mathbb{R}) = L^2(\mathbb{R}_*^-) \oplus L^2(\mathbb{R}_*^+)$, we can consider the representation $U^{(-)} \oplus U^{(+)}$ of G in $L^2(\mathbb{R})$; then, we set

$$\widetilde{U}(a, r) := \mathcal{F}((U^{(-)} \oplus U^{(+)})(a, r))\mathcal{F}^*, \qquad \forall\, (a, r) \in \mathbb{R} \rtimes \mathbb{R}_*^+. \tag{6.33}$$

For every $\psi \in L^2(\mathbb{R})$, we have

$$\psi_{a,r}(a') \equiv (\widetilde{U}(a, r)\, \psi)(a') = r^{-\frac{1}{2}}\, \psi((a' - a)/r), \qquad a, a' \in \mathbb{R}, \quad r \in \mathbb{R}_*^+. \tag{6.34}$$

Observe that this is the typical dependence on the translation and dilation parameters of a 'wavelet frame' (see [31]; note that the symbols that we use here for these parameters are non-standard). However, a function $\psi \in L^2(\mathbb{R})$, in order to be a 'good mother wavelet'—i.e. in order to verify the the orthogonality relations

$$\int_G \langle \phi, \psi_{a,r}\rangle\langle \psi_{a,r}, \phi\rangle\, d\mu_L(a, r) = \langle \phi, \phi\rangle, \qquad \forall\, \phi \in L^2(\mathbb{R}) \tag{6.35}$$

—has to satisfy suitable conditions. Indeed, as the reader will easily understand, one has to require that the following conditions hold:

(i) the projection onto $L^2(\mathbb{R}_*^\pm)$ (regarded as a subspace of $L^2(\mathbb{R})$) of the Fourier transform of ψ belongs to $\mathrm{Dom}(\hat{D}_{(\pm)})$, i.e.

$$\big(\mathbb{R}_*^\pm \ni x \mapsto |x|^{-1}\, |(\mathcal{F}\,\psi)(x)|^2\big) \in L^1(\mathbb{R}_*^\pm); \tag{6.36}$$

(ii) denoted by $\varepsilon_{\mathbb{R}_*^\pm}$ the characteristic function of the subset $\mathbb{R}_*^\pm$ of $\mathbb{R}$—observe that the orthogonal projection of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_*^\pm)$ is just the multiplication operator by $\varepsilon_{\mathbb{R}_*^\pm}$—the vectors

$$\hat{D}_{(-)}\big(\varepsilon_{\mathbb{R}_*^-}(\mathcal{F}\psi)\big) \in L^2\big(\mathbb{R}_*^-\big) \qquad \text{and} \qquad \hat{D}_{(+)}\big(\varepsilon_{\mathbb{R}_*^+}(\mathcal{F}\psi)\big) \in L^2\big(\mathbb{R}_*^+\big) \tag{6.37}$$

are both normalized, i.e.

$$2\pi \int_{\mathbb{R}_*^-} |x|^{-1}|(\mathcal{F}\psi)(x)|^2\, dx = 2\pi \int_{\mathbb{R}_*^+} |x|^{-1}|(\mathcal{F}\psi)(x)|^2\, dx = 1. \tag{6.38}$$

7. Conclusions, final remarks and perspectives

In this paper, we have considered star products from a purely group-theoretical point of view. In particular, we have not assumed to deal with Lie groups, but, in general, with locally compact topological groups. Therefore, our treatment allows us to include in a unified framework, for instance, all the finite groups (in the paper regarded as compact groups). This feature is certainly appealing in view of the increasing interest in realizing quantum mechanics on discrete spaces (see [45] and references therein). We think, in particular, that applying our results to a formulation of quantum mechanics on finite groups would be extremely interesting.

Let us briefly review the main points of our work. We have first recalled—see section 3—that with a square integrable (in general, projective) representation $U : G \to \mathcal{U}(\mathcal{H})$ of a locally compact group G are naturally associated a dequantization (Wigner) map $\mathfrak{S}_U$, which is an isometry, and its adjoint, the quantization (Weyl) map $\mathfrak{S}_U^*$. The standard Wigner and Weyl maps are recovered in the case where the group under consideration is the group of translations on phase space, up to a (symplectic) Fourier transform. We stress that this Fourier transform does not play any—mathematically or conceptually—relevant role; essentially, it allows us to obtain the usual quantization rule for the functions of position and momentum.

Next, in section 4, we have observed that by means of the quantization and dequantization maps associated with the representation U one can define a star product of functions enjoying remarkable properties. Endowed with this product and with a suitable involution, the Hilbert space $L^2(G)$ becomes a H^*-algebra $\mathcal{A}_U$, and—regarding G as a 'symmetry group' of a quantum system—the star product is, by construction, equivariant with respect to the natural action of G in $\mathcal{A}_U$, i.e. the action with which the standard symmetry action of G on states or observables in the Hilbert space $\mathcal{H}$ is intertwined via the Wigner map. Observe that the star product associated with U is such that the canonical ideal of $\mathcal{A}_U$—ideal which coincides with the range $\mathcal{R}_U$ of $\mathfrak{S}_U$—is a *simple* H^*-algebra (see [37, 38]), isomorphic to $\mathcal{B}_2(\mathcal{H})$. It is clear that the algebra $\mathcal{A}_U$ is commutative if and only if $\dim(\mathcal{H}) = 1$ (in this case, the square-integrability of U forces the group G to be compact). Observe moreover that, in the case where G admits various unitarily inequivalent *unitary* representations, one can define more general star products by forming suitable 'orthogonal sums' of 'simple' star products; see, e.g., formula (6.32). In section 4, we have also considered an interesting deformation of the star product associated with U, namely the $\hat{K}$-deformed star product, and studied its main properties. We will consider applications of this deformed product elsewhere.

At this point, our main task has been to derive explicit formulas for the previously defined star products. This task has been accomplished in section 5. We have shown that for every orthonormal basis contained in the domain of the positive self-adjoint operator $\hat{D}_U^{-2}$ (with $\hat{D}_U$ denoting the Duflo–Moore operator associated with U) one has a realization of the star product, see theorem 5.1. In the case where the group G is unimodular, the star product of two functions *belonging to the range of the Wigner map* $\mathfrak{S}_U$ assumes the particularly simple form of a 'twisted convolution', which reduces to the standard convolution if U is a unitary representation. It is interesting to note, incidentally, that it is the Banach space $L^1(G)$ which is usually endowed with the structure of a Banach $*$-algebra by means of convolution [27], while in $L^2(G)$ the convolution product is, in general, an 'ill-posed' operation. Namely, if the convolution product exists and belongs to $L^2(G)$ *for all pairs of functions* in $L^2(G)$, then the group G must be compact (recall, however, that by Hölder's inequality, the convolution of any pair of functions in $L^2(G)$ does exist, for G unimodular). This is a particular case ($p = 2$) of the classical 'L^p-conjecture' ($p > 1$), which has been finally proved (in its general form) in 1990 by Saeki [46]. Therefore, the *whole* vector space $L^2(G)$ can be endowed with the structure of an algebra by means of the convolution product if and only if G is compact.

Consider, now, the specific case where the group G is compact. In this case, one obtains a nice decomposition formula for the convolution in $L^2(G)$ in terms of the star products associated with a realization of the unitary dual $\check{G}$ of G; see corollary 5.2. The Hilbert space $L^2(G)$, endowed with the convolution product and with the involution (5.36), is a H^*-algebra which we denote by $\mathcal{L}(G)$. The orthogonal sum decomposition (3.9)—complemented by formula (5.35)—can be regarded as the decomposition into minimal closed (two-sided) ideals of $\mathcal{L}(G)$ prescribed by the 'second Wedderburn structure theorem for H^*-algebras' [37, 38]. Any of these ideals—say $\mathcal{R}_U = L^2(G)_{[U]}$—is a simple finite-dimensional H^*-algebra which is embedded, in a natural way, in the H^*-algebra $\mathcal{A}_U$ determined by the star product (5.47) and by the involution (5.36); precisely, as already observed, $\mathcal{R}_U$ is the canonical ideal of $\mathcal{A}_U$. It is actually the interest in the algebra $\mathcal{L}(G)$ that motivated Ambrose's study of H^*-algebras [37]. In our opinion, the formalism of star products provides a concrete and conceptually clear framework for Ambrose's ideas. Incidentally, note that the definition of a H^*-algebra given in section 4 may seem to be slightly stricter than the original definition given by Ambrose. However, it is easy to show that they are actually equivalent.

It is worth observing that—different from the quantization–dequantization scheme which has been recently developed in [23]—in the 'Weyl–Wigner approach' that is considered in the present contribution there is no canonical way for representing a *generic* quantum observable as a suitable 'phase space function' since, for $\mathcal{H}$ infinite-dimensional, $\mathcal{B}_2(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H})$ (in the case of the standard Weyl quantization, this problem has been studied, for instance, in [47]).

This feature, of course, reflects in the fact that there is no standard way for representing within the framework considered here the product of a *generic* quantum observable by a state as a star product of functions. However, we believe that suitably extending the domain of the first argument of the star product—this time *defined* as the rhs of (5.26)—from $L^2(G)$ to some larger space of functions (or distributions), and, possibly, restricting the domain of the second argument, one should be able to generalize the results obtained in the paper. This interesting topic will be the object of further investigation.

One can, in principle, elaborate several examples of star products defined along the lines traced in the present paper that are potentially relevant for applications. In addition to the case of compact groups, for all groups admitting square integrable projective representations, it is possible to define star products of functions. In section 6, we have considered the significant examples of the group of translations on phase space and of the affine group, but, of course, several other examples would deserve attention. As an example, we mention the group $SL(2, \mathbb{R})$. According to classical results due to Bargmann [48], this group admits a (infinite) countable set of mutually unitarily inequivalent, square integrable unitary representations—the 'discrete series'—with carrier Hilbert spaces consisting of suitable holomorphic functions on the upper half plane.

A wide class of groups with important applications in physics and related research areas (in particular, signal analysis) is formed by the semi-direct products with an Abelian normal factor. For these groups square integrable representations can be suitably characterized, see [44], and examples of such groups, admitting square integrable representations and having remarkable applications, can be found in [24, 25]. From the point of view of signal analysis, the image through the Weyl map of a function in $L^2(G)$ can be regarded as a *localization operator* of a different kind with respect to the localization operators usually considered in wavelet and Gabor analysis [31]. Thus, the star product provides a way for characterizing the product of two localization operators. Possible applications of our results to signal analysis is a further topic that we plan to investigate in the future.

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